The Structure of Commutayive Banach Algebra

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ABSTRACT: Let A be a Banach algebra over C with norm $\|.\|$. In this note, several characterizations of commutativity of A are given. For instance, it is shown that A is commutative if

$$\|AB\| = \|BA\|, \forall A, B \in A$$

or if the spectral radius on A is a norm.

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INTRODUCTION

Functional analysis is the study of differentiation, integration, estimates and asymptotic of functions of real numbers. This is the modern research version of calculus.

Functional analysis is the study of certain topological algebraic structures and of the methods by which knowledge of these structures can be applied to analytic problems.

The theory of Banach algebra (BA) is an abstract mathematical theory which is the (sometimes unexpected) synthesis of many specific cases from different areas of mathematics.

Banachalgebra are rooted in the early twentieth century, when abstract concepts and structures were introduced, transforming both the mathematical language and practice. In the 1930's general topology has been quite developed while functional analysis was evolved through the Hahn-Banach theorem.

The uniform bounded theorem, the theorem of closed graph and the open mapping theorem, all of them are 1932-theorems of Banach, whose book, **"THEORIE DES OPERATIONS LINEARIES (1932)",** influenced deeply mathematical analysis of his era.

Riesz provides for the first time the axioms for a space with a norm $\|.\|$ in 1918. Banach'sthesis, first abstract study of normed spaces in 1920.von Neumann studies additional structure on a normed spaces in 1929.

Stone's **"LINEAR TRANSFORMATIONS IN HILBERT SPACE AND THEIR APPLICATIONS TO ANALYSIS"** is a major contribution to operator theory.

Gelfand-Naimark representation theorems in 1943.Naimark's book **"NORMED RINGS"** is the first presentation of the whole new theory of Banach algebra, which was important to its development.

Rickart's book **"GENERAL THEORY OF BANACH ALGEBRAS"** is the reference book of all the later studies of Banachalgebra.A branch of analysis which studies the properties of mappings of classes of function from one topological vector space to another. This subject is huge and growing rapidly.

Functional analysis is a complex blend of algebra and topology with its evolution influenced by the development of these two branches of mathematics.

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We shall study a new algebraic structure is called a banach algebra. **“THE STRUCTURE OF COMMUTATIVE BANACH ALGEBRA”** is interesting topics in functional analysis and develops necessary general preliminaries for the study of Gelfand theory of banach algebra.

PRELIMINARIES

Definition 1.1

A **Linear space** over a field F is a set v with the operation called vector addition defined on $V \times V \to V$ given by $(x, y) \to x + y$ and an operation called vector multiplication defined on $F \times V \to V$ given by $(\alpha, x) \to \alpha x$ satisfying the following conditions $\forall x, y, z \in V$ and $\alpha, \beta \in F$.

i) $(x + y) + z = x + (y + z)$

ii) $x + y = y + x$

iii) there exists an element $0 \in V$ such that $x + 0 = 0 + x = x$

iv) for each $x \in V$ there exists an element $-x \in V$ such that $x + (-x) = (-x) + x = 0$

v) $\alpha(x + y) = \alpha x + \alpha y$

vi) $(\alpha\beta)x = \alpha(\beta x)$

vii) $(\alpha + \beta)x = \alpha x + \beta x$

viii) $1.x = x$

Definition 1.2

A **Normed linear space** is a linear space in which each vector $\bar{x}$ corresponds to a real number defined by $\|x\|$ called the norm of x in such a way that,

i) $\|x\| \geq 0 and \|x\| = 0 \Leftrightarrow \bar{x} = 0$

ii) $\|x + y\| \leq \|x\| + \|y\|$

iii) $\|\alpha x\| = |\alpha| \, \|x\|$

Definition 1.3

A complete normed linear space is called **Banach space**.

Example 1.3.1

1. R is a banach space.
2. C is a banach space.

Definition 1.4

A is a Banach space with respect to norm that satisfies the Multiplicative inequality,

$$\|xy\| \leq \|x\| \, \|y\|, \ \forall x, y \in A$$

And if A contains a unit element e such that,

$$xe = ex = x, \ \forall x \in A \text{ and } \|e\| = 1.$$

Then A is called **Banach algebra**.

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Definition 1.5

A **topology** on a set x is a collection τ of subsets of x having the following properties,

i) ϕ and x are in τ .

ii) The union of the elements of any subcollection of τ is in τ .

iii) The intersection of the elements of any finite subcollection of τ is in τ .

A set x for which topology τ has been specified is called **Topological space**.

Note

If X is a set and a topology τ on X the topological space is denoted by (X,τ).

GELFAND THEORY OF COMMUTATIVE BANACH ALGEBRA

Definition 2.1

Banach algebra A is said to be **commutative banach algebra** if

$$xy = yx, \ \forall x, y \in A$$

Definition 2.2

Let Δ be the set of all complex homomorphism of commutativebanach algebra A. The formula,

$$\hat{x}(h) = h(x), \ (h \in \Delta)$$

Assigns to each $x \in A$ a function $\hat{x} : \Delta \to C$ we call $\hat{x}$ be the**Gelfand transform** of x.

Definition 2.3

Let $\hat{A}$ be the set of all $\hat{x}$, for $x \in A$.

The **Gelfand topology** of Δ is the weak topology induced $\hat{A}$, that is the weak topology that makes every $\hat{x}$ iscontinuous.

Then obviously $\hat{A} \subset C(\Delta)$ the algebra of all complex continuous functions on Δ.

Theorem 2.6

If A is a commutative Banach algebra and

$$r = \inf \frac{\|x^2\|}{\|x\|^2} \text{ and } s = \inf \frac{\|\hat{x}\|_\infty}{\|x\|}$$

Where $x \in A, x \neq 0$, then $s^2 \leq r \leq s$

Proof:

Let A is a commutative Banach algebra.

To prove:

$$s^2 \leq r \leq s$$

For each $x \in X$ the null space of $\hat{x}$ in $\sigma(x)$.

Thus $\|\hat{x}\|_\infty \geq s\|x\|, \forall x \in A$----(1)

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For each $x \in A$, the range of $\hat{x}$ in $\sigma(x)$.

Hence,

$$\|\hat{x}\|_\infty = \rho(x) \le \|x\|$$

$$\|\hat{x}\|_\infty \le \|x\|$$

$$\|x\| \ge \|\hat{x}\|_\infty$$

Squaring on both sides, we get

$$\|x^2\| \ge \|\hat{x}^2\|_\infty = \|\hat{x}\|^2{}_\infty \ge s^2\|x\|^2$$

$$\|x^2\| \ge \|\hat{x}\|_\infty^2 \ge s^2\|x\|^2,\ \forall x \in A$$

$$\|x\|^2 \ge s^2\|x\|^2$$

$$\frac{\|x^2\|}{\|x\|^2} \ge s^2$$

$$r \ge s^2 \quad \text{------(2)}$$

Nextwe have to prove that,

$$r \le s$$

For each $x \in X$ the range of $\hat{x}$ in $\sigma(x)$.

Thus,

$$\|x^2\| \ge r\|x\|^2,\ \forall x \in A$$

By induction hypothesis on ‘n’ shows that

$$\|x^m\| \ge r^{m-1}\|x\|^m$$

where $m = 2^n$, n=1,2,3… ------(3)

Step: 1

Put m=2 in equation (3), we get,

$$\|x^2\| \ge r\|x\|^2$$

Hence the result is true for (3).

Step: 2

Let us assume that the result is true for m.

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Step: 3

Let the result is always true for $m+1$.

Take the m^{th} roots in equation (3) and let $m \to \infty$

Let Δ be the maximal ideal space of a commutative Banach algebra A.

Hence,

$$\|\hat{x}\|_{\infty} = \rho(x) \le \|x\|, \forall x \in A$$

$$\|\hat{x}\|_{\infty} = \rho(x) \ge r\|x\|$$

$$\|\hat{x}\|_{\infty} \ge r\|x\|$$

$$\frac{\|\hat{x}\|_{\infty}}{\|x\|} \ge r$$

$$s \ge r$$

$$r \le s \qquad \text{------(4)}$$

From equation (2) and (4)

$$s^2 \le r \text{ and } r \le s$$

Hence,

$$s^2 \le r \le s$$

Hence the proof.

INVOLUTIONS IN COMMUTATIVE BANACH ALGEBRA

Definition 3.1

A Banach algebra is called **Banach*-algebra**, if there exists a mapping $x \to x^*$ of A into itself called involution satisfying the following conditions.

1. $(x+y)^* = x^* + y^*$
2. $(\alpha x)^* = \overline{\alpha} x^*$, where $\overline{\alpha}$ is the complex conjugate of α
3. $(xy)^* = y^* x^*$
4. $x^{**} = x$, for all $x, y \in A \,\&\, \alpha \in C$

Note

The element x* is called the adjoint of x.

From the definition of involution, we note the following properties

1.$x \to x^*$ is a one-to-one mapping of A onto itselfif $x \ne y$ then from the of involution $x^* \ne y^*$

2.$0^* = 0$

$$0 + x^* = x^*$$

$$= (0 + x)^*$$

$$= 0^* + x^*$$

$$0 = 0^*$$

$$3. e^* = e$$

For, $e^* = ee^*$

$$= e^{**} e^*$$

$$= (ee^*)^*$$

$$= (e^*)^*$$

$$= e^{**}$$

Hence,

$$e^* = e$$

Definition 3.2

Let A and A^1 be Banach*-algebra. Then the isomorphism from A onto A^1 is called *** isomorphism** if it preserves involution in the sense that,

$$f(x^*) = f(x)^*.$$

Definition 3.3

A Banach *-algebra A is called a **B*-algebra,** if it the following relation between involution and the norm exists.

$$\|x^* x\| = \|x\|^2, \ \forall x \in A.$$

Note

$$\|x^* x\| = \|x\|^2 = \|x\| \ \|x^*\|, \ \forall x^* \in A, x \in A.$$

Theorem 3.8

If A is B*-algebra, then $\|x^*\| = \|x\|$.

Proof:

From the definition of B *-algebra, we get,

$$\|x\|^2 = \|x^* x\|$$

$$\|x\|^2 \le \|x^*\| \ \|x\|$$

$$\|x\| \le \|x^*\|, \ \forall x \in A \qquad \text{----------(1)}$$

Put $x = x^*$, we get,

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$\|x^*\| \leq \|(x^*)^*\|$

$= \|x^{**}\|$

$= \|x\|$

Hence,

$\|x^*\| \leq \|x\|,\ \forall x^* \in A, x \in A.$ ---------(2)

From (1) and (2), we get,

$\|x^*\| = \|x\|,\ \forall x \in A.$

Hence the proof.

CONCLUSION

In this dissertation discusses the Commutative Banach Algebra. It reveals that a close interplay between the theory of Gelfand theory of commutative banach algebras. It presents some basic facts about Commutative Banach algebras. It defines the spectrum of an element of a banach algebra and establishes its basic properties.

This chapter presents a theorem in which each non-zero element is invertible. Then A is isometrically isomorphic to . This chapter is devoted to the study of ideals and homomorphism's of commutative banach algebras. It is devoted to the study of the joint spectrum of elements of a commutativebanach algebra.

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