A Study on Quasi - Groups Satisfying Certain Identities and Abelian Quasi Groups

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Abstract:- A quasi- group is a group like structure(Q,*) which satisfies the latin square property, but neednot have an identity element, nor need it be associative. It coincides with the notion of a divisible magma. In this paper we make a study of quasi- groups which satisfy certain identities and Abelian Quasi- groups.

Keywords:- *Magma, latin square property, principal unit, normal sub-quasi-group.*

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I. INTRODUCTION

A theory of non- associative algebras have been developed without any assumption of a substitute for associative law,and the basic structure properties of such algebras have been shown to depend upon the possession of almost the same properties by related associative algebras.

In this paper we derive certain structural properties of quasi-groups that are mentioned in B.A. Hausmann and O.Ore [6], D.C.Murdoch [7]. In the second section we make a study on quasi-groups which satisfy the law

(ab) (cd) =(ac) (bd), known as Abelian Quasi-group(Associative law III).

This equation is also known as the functional equation of bisymmetry on a Quasi-group.

II. PRELIMINARIES

Definition 2.1. A quasi-group G is a set together with an operation of multiplication such that

1) the set is closed under multiplication.
2) The equations ax = b and ya = b have unique solutions for x and y ,where a and b are any two (not necessarily distinct) elements of G

Condition (2) is sometimes referred to as quotient axiom or latin square property. The quotient- axiom (2) implies both left and right cancellation laws .Since we are considering only finite quasi- groups, it is useful to note that every subset of G which isclosed under multiplication satisfies the quotient- axiom and is therefore a sub-quasi group of G. From the quotient- axiom (2) it follows that every element a in G has a right unit e_a and a left unit e_a'defined by

$ae_a = e_a'a = a$

Definition 2.2. If one of the minimal right unit sub-quasi group of a quasi-group G consists of a single element e, then e will become its own right (and left) unit. In this case e will be called the principal unit.

Also the set H of all principal units of G forms a sub-quasi-group of G.

Definition 2.3. Let H be a sub-quasi-group of G, then if

$$H\, f_a\, (bH) = b\, f_a\, (H), \text{ for all } a, b \in H$$

then H is called a left-normal-sub-quasi-group of G.

The following are few examples of a quasi - group.

1. The set of integers Z under the binary operation, subtraction (-) forms a quasi- group.

2. The non- zero rational numbers (or non- zero real numbers) with division forms a quasi- group.

III. QUASI – GROUPS SATISFYING CERTAIN IDENTITIES

We denote by O the null sub-quasi-group containing no elements and contained in every sub-quasi-group of G. Then the sub-quasi-groups of G will form a lattice. In this paper we investigate the condition under which the normal-sub-quasi-groups form a lattice. We also prove the Jordan-Holder theorem for quasi-groups.

Definition 3.1 :-

Two sub-quasi-groups H and K of G are said to be permutable if for any two elements h and k,

$$hk = k'h', \text{ where } k' \in K \text{ and } h' \in H.$$

Theorem : 3.1

Any two normal sub-quasi-groups H and K which have a non –void cross cut D are permutable.

Proof :-

Let h, k be any two elements of H and K respectively and d be any fixed element of D.Then

hk = (dh′) (k′d)

= d[h′f$_d$ (k′d) (by normality of H and k)

= d[k′ f$_d$ (h′)]

= (dk′) h′

= k′ h′ where k′∈ k & h′∈H

Also kh = (dk′) (h′d)

= d[k′ f$_d$ (h′d) [by associative law I]

= d[h′ f$_d$ (k′)]

= (dh′) k′

= h′ k′

Thus hk = k′h′ and kh = h′ k′.

∴H and K are permutable.

Hence the theorem .

Remark : - If the cross cut (H,K) is non –empty, and if H and K are normal, then the union [H,K] consists of all elements, and only those if the form hk.

Theorem 3.2:

The Jordan – HolderTheorem:-

The normal sub-quasi-groups of G which contain the right unit R form a Dedekind structure .

Proof:-

Let H and K be two normal sub-quasi-groups of G. First we have to show that they form a structure. In order to show that they form a lattice it is sufficient to show that union [H,K] is normal.

Now for a sub-quasi-group H containing R the normality condition reduces to

(1) H(bH) = f^{-1}_a (b)H ,For all elements a and b in G.

From theorem3.1, it follows that

[H,K] = HK.

Also from the normality of H and K and theorem 3.1 we have

(HK) [b (Hk)] = (KH) [(bH)K]

= [(KH) (bH)] K

= [(Kb)H]K

= K[(bH)K]

= f^{-1}_a (bH) K

= f^{-1}_a (b) (HK)],

for all a and b in G.

Hence [H,K] is normal, and the normal sub-quasi-group containing R from a lattice.

To show that it is a Dedekind structure, it is necessary to show that if M is any elements containing H, then

(M, [H, K]) = [H, (M,K)]

Now

(2) [H, (M,K)] ⊆ (M, [H,K])

Since M ⊃ H and [H,K] = {hk/h ∈ H, k ∈ k}, it follows that

(3) (M, [H, K]) ⊆[H, (M,K)]

From (2) and (3) we get

(M, [H,K] = [H, (M,K).

Hence the theorem.

The Dedekind structure will contain a unit element $\bar{R}$ and if R is normal, R =$\bar{R}$. It is shown by Ore [6] that all principal chains of normal sub-quasi-groups between G and =$\bar{R}$ have the same length. Further, the quotient structures between successive terms in any one such principal chain are isomorphic in some order to those in any other.

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Theorem 3.3:-

If H and K are any two permutable sub-quasi-groups of G which contain R, and if H is normal in the union [H,K], then the cross cut (H,K) is normal in K and

$[H,K]/H \cong K/(H,K)$.

Proof:-

Given H is normal in [H,K] = HK.

ie, if hk, $h'k' \in$ HK then

(4) $[(hk)H]\,[(h'k')H] = (hkh'k')\,H$

Let $k_1, k_2 \in K$ and put $L = H \cap K$

Now $(k_1 l_1)(k_2 l_2) \in (k_1 L)(k_2 L)$

Where $k_1, k_2 \in K$ and $l_1, l_2 \in L = H \cap K$

(5) $(k_1 l_1)(k_2 l_2) = [(k_1 e_{k1}) l_1]\,[(k_2 e_{k2}) l_2]$

Also $e_{k1}, e_{k2} \in R \subset H$ by hypothesis.

$[(k_1 e_{k1}) l_1]\,[(k_2 e_{k2}) l_2] \in [(k_1 e_{k1})H]\,[(k_2 e_{k2})H]$

$[\because l_1, l_2 \in H \cap K \subset H]$

ie, $[k_1 e_{k1}) l_1]\,[(k_2 e_{k2}) l_2] \in (k_1 H)(k_2 H) = (k_1 k_2) H$ by (4).

L.H.S. of (5) is an element of K, since $k_1, k_2 \in K$ and.

$l_1, l_2, \in L \subset K$.

$\therefore$(5) can be written as $k_3 = k_1, k_2\, h$.

$\Rightarrow h \in K, \because h \in H, h \in H \cap K = L$.

(6) $\therefore (k_1 L)(k_2 L) \subset (k_1 k_2) L$

If $(k_1 k_2) l \in (k_1 k_2) L$

Then $(k_1 k_2) l = (k_1 e_{k1})(k_2 l) \in (k_1 L)(k_2 L)$

Since $e_{k1} \in R \subset H \cap K = L$, by hypothesis $R \subset H$ and K

(7) $\therefore (k_1 k_2) L \subset (k_1 L)(k_2 L)$

(6) and (7) gives

$(k_1 k_2) L = (k_1 L)(k_2 L)$.

$\therefore L = H \cap K$ is normal in K.

ie, (H,K) is normal in K.

Also if $\phi : G \to G'$ is a homomorphism with Kernel K, then G/K is isomorphic to G' in groups. This result holds for quasi- groups also.

Define ϕ: HK $\to$ K/H $\cap$K by

$\phi(hk) = h\,(H \cap K)$

Then $K_\phi = \{hk : \phi(hk) = H \cap K\}$

Appling the above result,

$HK/H \cong K/H \cap K$.

Hence the result.

IV. ABELIAN – QUASI – GROUPS :-

In this section we consider quasi-groups satisfying the following equation. Let G be a quasi – group and a,b,c,d $\in$ G. Then we define G an Abelian-quasi-group if

(ab) (cd) = (ac) (bd).

We call the above equation, the Associative Law III.

If a is any element of G, any element of the cyclic – quasi –group generated by a, will be called the power of a, and is denoted by $\phi_r(a)$ where r is the number of factors a which occur. We can then prove the following theorem which generalises the law

$(ab)^r = a^r b^r$ of Abelian groups.

*Theorem*4.1:-

It a and b are any two elements of an Abelian-quasi-group, and if $\phi_n(a)$ is any power of a, then

$\phi_n(ab) = \phi_n(a)\phi_n(b)$.

Proof:-

Suppose n = 2; then

$\phi_2(ab) = (ab)^2$

$= (ab)(ab)$

$= (a\,a)(b\,b)$ [$\because$ G is an abelian quasi-group]

$= a^2 b^2$

$= \phi_2(a)\,\phi_2(b)$.

Clearly the theorem holds for n =2.

Assume that the theorem is true for all powers r<n. Since every power $\phi_n(a)$ can be written as the product of two such powers, for some r less than n we have

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ϕ_n (ab) $=\psi_r(ab)\chi_{n-r}(ab)$

$= [\psi_r(a)\ \psi_r\ (b)]\ [\chi_{n-r}(a)\ \chi_{n-r}\ (b)]$

$= [\psi_r\ (a)\ \chi_{n-r}\ (a)\ ]\ [\psi_r\ (b)\chi_{n-r}\ (b)]$

[$\because$G is an abelian – quasi – group)

$= \phi_n\ (a)\ \phi_n\ (b)$.

$\therefore \phi_n\ (ab) = \phi_n(a)\ \phi_n(b)$.

The theorem holds for power with n factors, and therefore holds in general.

Corollary :-If ϕ_n (a), ψ_m (a) are any two powers of a, then

$$\phi_n\ [\psi_m\ (a)] = \psi_m\ [\phi_n\ (a)].$$

Proof :-

We have

$$\phi_n\ (ab) = \phi_n(a)\ \phi_n\ (b).$$

Putting b = a in the above equation, we have

$$\phi_n\ (a)^2 = [\phi_n(a)]^2.$$

Then

$$\phi_n\ [\psi_m(a)] = \phi_n\ [a^m]$$

$$= [\phi_n\ (a)]^m$$

$$= \psi_m\ [\phi_n\ (a)].$$

$$\therefore \phi_n\ [\psi_m(a)] = \psi_m\ [\phi_n\ (a)].$$

Hence the result.

Theorem 4.2 :-

In an Abelian – quasi-group, the set of all sub-quasi-groups which contain a given minimal unit sub-quasi-group, forms a Dedekind structure.

Proof :-

First we have to prove that in an abelian – quasi-group all sub – quasi-groups are normal.

Let $ah_1 \in aH$, $bh_2 \in bH$

$(ah_1)\ (bh_2) = (ab)\ (h_1\ h_2)$ ($\because$G is abelian)

$= (ab)\ h_3 \in (ab)H$ where $h_3 = h_1\ h_2$.

$\therefore (aH)\ (bH) = (ab)H$.

$\therefore$ In an abelian-quasi-group all sub-quasi-groups are normal. Also G contains a minimal unit sub-quasi-group. From theorem 3.2it follows that the normal sub-quasi-groups of G which contain right unit R form a Dedekind structure.

Hence the theorem.

Theorem 4.3:-

If G is Abelian, the quotient quasi-group G/R has a unique right unit R. Every left coset of R is also a right coset of R. The mapping aR→ Ra is an automorphism of G/R and is equivalent to left multiplication by the right unit R. Finally if aR = Ra for all a in G then G/R is a group.

Proof :-

We have

$(aR)\ R = a\ [R\ f_a\ (R)] = aR$ for all $a \in G$.

This implies that R is a right unit of G/H.

Since G/R is a quasi-group, R is unique. Next we have to show that

ϕ: aR→ Ra is an automorphism.

Since $(aR)\ \phi = (bR)\ \phi \Rightarrow aR = bR$, ϕ is one – one.

ϕ is a homorphism, for,

$[(aR)\ (bR)]\ \phi = [(ab)\ R]\ \phi$

$= Rab$

$= (Ra)\ (Rb)$

$= (aR)\phi\ (bR)\phi$.

Also since ϕ is onto, it is an automorphism.

From equation (1) of theorem 3.2it follows that

$R\ (cR) = (cR)^s = f^{-1}_a\ (c)\ R$, for all a and c in G.

Putting $a = e_a$ and $d = e_c$ in associative law III we find

$f^{-1}_a\ (b)c = f^{-1}_a\ (c)\ (be_c)$

Let b run through all elements of R and we have

$Rc = f^{-1}_a\ (c)\ R$ and therefore

$R\ (cR) = (cR)^s = Rc$.

Hence the automophism ϕ : aR → Ra is equivalent to the left-multiplication by the right unit R.

To show that G/R is a group if aR = Ra, for all $a \in G$.

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Since G is a quasi-group and a, b, $\in$ G $\Rightarrow$ ab $\in$ G.

Let aR, bR$\in$ G/R

Then we have

$$(aR)\ (bR) = (ab)\ R \in G/R.$$

Thus G/R is closed under multiplication.

Further

$$R\ (cR) = Rc = cR.$$

Also (cR) R = c[R f_a (R)]= cR

$\therefore$R (cR) = cR = (cR)R.

$\therefore$R is the identity in G/R

Also

$$[(aR)\ (bR)]\ (cR) = [(aR)\ (bR)]\ [R\ (cR)]$$
$$= [(aR)\ (bR)]\ [(a_e\ R)\ (cR)]$$
$$= (aa_e)\ R\ [(bR)\ (cR)]$$
$$= (aR)\ [(bR)\ (cR)]$$

$$\therefore [(ab)\ c]\ R = [a\ (bc)]\ R.$$

Associativity holds in G/R.

Hence if aR = Ra for all a in G then G/R is a group.

Hence the theorem.

Also we can show that the set H of all elements which commute with the unit element e, is a group, the largest group contained in G.

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