Contributions from Homology Theory

Satish Kumar Tiwari[1] and Bijoy Kumar Singh[2]
[1]Veer Kunwar Singh University Ara, Bihar, India.
[2]Jain College Ara, Bihar, India.

Abstract: This paper deals with the different contributions from Homology theory. It defines homology in homotopy type theory. The author has provided proofs for many theorems and explained the theory giving suitable examples.

Keywords: *Homology theory, abelian group, sub-module, morphism, etc.*

*****

I. INTRODUCTION

1.1 Singular Homology

Let us return to topololgy to construct the (singular) homology functions H_n : Top $\to$ Ab, one function for each $n \geq 0$; we provide more details for a generalization of our earlier discussion of curves in planar regions.

For each $n \geq 0$, consider euclidean n-space R^n imbedded R^{*1} as all vectors whose last coordinate is 0. Let v_0 denote the origin, and let $\{v_1, v_2, ..., v_n\}$ be the standard orthanormal basic of R_n (v_i has 1 in the ith coordinate and 0 elsewhere). For each $n \geq 0$, let $\Delta_n = \{(t_1, ...t_n)\ t_i \geq 0$ all i, and $\sum t_i = 1\}$ be the convex set spanned by $\{v_0, ...\ v_n\}$; Δ_n is called the standard n-simplex with vertices $\{v_0, ..., v_n\}$ and is also denoted $\Delta_n = [v_0, ..., v_n]$. Thus, $\Delta_0 = [v_0]$ is a point, $\Delta_1 = [v_0, v_1]$ is the uniinterval [0, 1]; $\Delta_2 = [v_0, v_1, v_2]$ is the triangle (with interior) having vertices $v_1, v_2,$; Δ_3 is a tetrahedron, and so forth. A curve in a topological space X is a continuous map $\sigma : \Delta_t \to X$; a closed curve in X is a curve σ with $\sigma(N) = \sigma(1)$. The boundary of Δ_t is {0, 1}; more generally, the boundary of Δ_n is $\cup_{i=0}^{n}$ $[v_0, ..., \hat{v}_i ,.......\ v_n]$, where means "delete". However, we need an "oriented boundary" if we are to generalize the picture of Green'stheorem [1-3].

1.2 Definition : An orientation of Δ_n is an ordering of its vertices [4].

It is clear that different orderings may give the "same" orientation. For example, consider Δ_2 with its vertices ordered $v_0 < v_1 < v_2$. A tour of the vertices shows that Δ_2 is oriented counterclokwise. Thus, the orderings

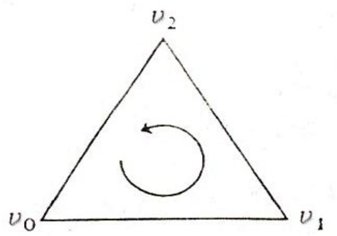

Figure (x)

$v_1 < v_2 < v_0$ and $v_2 < v_0 < v_1$ give the same tour, while the other three permutations give a clockwise tour.

1.3 Definition. Two orientation of Δ_n are the same if, as permutations of $\{v_0,...., v_n]$, they have the same parity (both are even or both are odd); otherwise, the orientation are opposite [5].

After examining this definition for the tetrahedron Δ_3, the reader will be content with it.

Given an orientation of Δ_n, one may orient $[v_0.......\ \hat{v}_i ,\ ......\ v_n]$ in the sense $(-1)^i\ [v_0.......\ \hat{v}_i ,\ ......\ v_n]$, where $-[v_0.......\ \hat{v}_i ,\ ......\ v_n]$ means its orientation is opposite to that of $[v_0.......\ \hat{v}_i ,\ ......\ v_n]$ (vertices in displayed order). For example, consider Δ_2 oriented counter clockwise:

 ISSN: 2454-4248
53 – 60

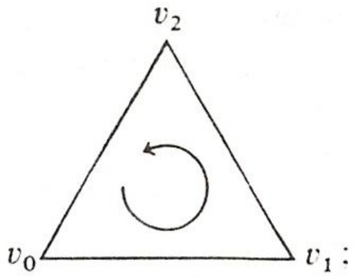

Figure (xi)

the natural way to orient the edges is :

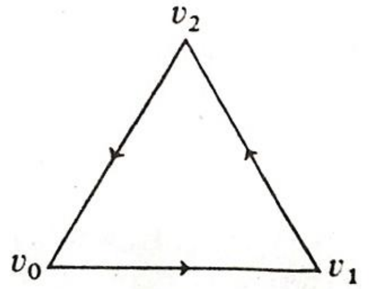

Figure (xii)

The edges are thus oriented $[v_0, v_1]$, $[v_1, v_2]$, and $[v_2, v_0]$. Since $[v_2, v_0] = -[v_0, v_2]$, the oriented boundary of Δ_2 is $[v_1, v_2] \cup -[v_0, v_2] \cup [v_0, v_1] = [v_0, v_1, v_2] \cup [v_0, \hat{v}_1, v_2] \cup [v_0\, v_1, \hat{v}_2]$. The oriented boundary of Δ_n should thus be $\cup_i^n = 0\ (-1)^i\, [v_o, v_1, \ldots\ldots \hat{v}_i, \ldots\ldots v_n]$.

1.4 Definition : If X is a topological space an n-simplex in X is a continues function $\sigma : \Delta_n - X$. The n-chains in X comprises X, the free abelian group with basis all n-simplexes in X. For convenience set $S_{-1}(X) = 0$.

Observe that $S_1(X)$ is precisely the group of chains suggested by line integrals: all formal linear combination of curve in X. The group $S_n(X)$ is the n-dimensional generalization of $S_1(X)$ we anticipated.

If $\sigma : \Delta_n \to X$, its boundary should be $\partial\sigma$ should be $\sum_{i=0}^{n} (-1)^{i}_{\sigma}\ v_{0_1} \ldots\ldots \hat{v}_i \ldots\ldots v_n]$. A technical problem arises. It would be nice if or were an (n– 1) chain : It is not because the domain of σ $[v_0 \ldots\ldots \hat{v}_i \ldots\ldots v_n]$ is not the standa (n – 1) simplex Δ_{n-1}. To state the problem is to solve it. For each i, define $e_i : \Delta_{n-1} \to \Delta_n$ as the affine map sending the lemses $\{v_0 \ldots\ldots, v_{n-1}\}$ to t vertices $\{v_0, \ldots\ldots \hat{v}_i \ldots\ldots v_n\}$that preserves the displayed orderings :

$$e_i\,(t_1 \ldots\ldots t_{n-1}) = t_1\ldots\ldots, t_{i-1}, 0, t_i - t_{n-1}) \in \Delta_n.$$

1.5 Definition. If $\sigma : \Delta_n \to X$, then $\partial_n\sigma = \sum_{i=0}^{n} (-1)^i\, \sigma e_i \in S_{n-1}(X)$.

II. THEOREMS

2.1 Theorem : There is a unique homomorphism dn: $S_n(X) \to S_{n-1}(X)$ with $\partial_n\sigma = \sum_{i=0}^{n} (-1)^i\, \sigma e_i$ for every n-simplex σ in X.

Proof :

The homomorphism ∂_n are called boundary operators; usually one omitted the subscript n.

We now have a sequence of homomorphism.

$$\ldots \to Sn(X) \underset{\to}{i_n} S_{n-1}(X) \to \ldots. \to S_1(X) \underset{\to}{i_1} S_0(X) \to 0.$$

Let us denote e_1: $\Delta_{n-1} \to \Delta_n$ by $[v_{0,} \ldots\ldots \hat{v}_i \ldots\ldots v_n]$.

 ISSN: 2454-4248
53 – 60

Lemma : *The following formulas hold* ……… e_j: $\Delta_{n-2} \to \Delta_n$: if $i < j$ then e_i o e_j = $[v_0$ ……, $\hat{v}_i$,……., $\hat{v}_j$, …… $v_n]$: if $i \geq j$, then e_i o ej = $[v_0$, …. $\hat{v}_j$, ….., $\hat{v}_{j+1}$, …… $v_n]$

Proof. The maps e_i and e_j, hence their composite, are completely determined by their values on the vertices $\{v_0$, …. $v_{i-2}\}$. The computation showing that the two displayed vertices are the deleted ones is routine.

2.2 Theorem : For each $n \geq 1$, we have $\partial_{n-1} \partial_n = 0$

Proof. It suffices to show $\partial\partial\sigma = 0$ for every n-simplex $\sigma : \Delta_n \to X$

$$\partial\partial\sigma = \partial\,(\Sigma(-1)^i\,\sigma e_I) = \Sigma\,(-1)^i\,\partial(\partial\sigma e_i) = \sum_{l+j} (-1)^{i+j}\ \sigma e_i e_j$$

$$= \sum_{i<j} (-1)^{i+j}\ \sigma\,[v_0, \ldots\ldots \hat{v}_i, \ldots\ldots, \hat{v}_j \ldots\ldots v_n]$$

$$+ \sum_{i>j} (-1)^{i+j}\ [v_0, \ldots\ldots \hat{v}_i, \ldots\ldots, \hat{v}_{i+1}, \ldots\ldots v_n]$$

Lemma : Now change variable in the second sum; set l = i and k = i +1. The second sum reads $\sum_{i<k} (-1)^{k+1-1}$ σ $[v_0$, ……. $\hat{v}_1$,……. $\hat{v}_k$, ….., $v_n]$. It is now clear that each term in that sight sum occurs in the left sum with opposite sign. Therefore all cancels and $\partial\partial\sigma = 0$.

2.3 Definition : An n-cycle is an elements of ker∂n; write ker∂n = $Z_n(X)$. An n-boundary is an element of im∂_{n+1}: write im∂_{n+1} = $B_n(X)$.

Both $Z_n(X)$ and $B_n(X)$ are subgroups of $S_n(X)$. Our discussion of the oriented boundary of Δ_n should make the definition of $\partial_n\sigma$ appear reasonable. It is also reasonable n-cycles are generalizations of closed curves; at the very least, this is so when n = 1. Assume σ: $\Delta_1 \to X$ is a closed curve, so that $\sigma(0) = \sigma(1)$. Now a 0-simplex in X can be identified with a point of X, so that $S_0(X)$ is the group of all formal linear combinations of the points of X. Furthermore, $\partial_1\sigma = \sigma(1) - \sigma(0)$ so a closed curve is a 1-cycle. As a second example, assume ρ, σ and τ are curves forming a triangular path in X : say, $p(0) = x_0$, $p(1) = x_1 = \sigma(0)$, $\sigma(1) = x_2 = \tau(0)$, and $\tau(1) = x_0$. Then $\partial_1\,(\rho + \sigma + \tau) = (x_1 - x_0) + (x_2 - x_1) + (x_0 - x_2) = 0$, and $\rho + \sigma + \tau$ is a 1-cycle [6&7].

Corollary : For each $n \geq 0$, we have $B_n(X) \subset Z_n(X) \subset S_n(X)$

Proof : If $\beta \in B_n(X)$, then $\beta = \partial y$ for some $y \in S_{n+1}(X)$. Thus $\partial\beta = \partial\partial\gamma = 0$, by Theorem 2.2, whence $\beta \in \ker\partial_n = Z_n(X)$.
Once we recall that Green's theorem tells us boundaries should be trivial, the next definition is forced on us.

2.4 Definition : The nth homology group of X is

$$H_n(X) = Z_n(X) / B_n(X).$$

The next few pages should be read without pausing to verify any particular assertion; more details will be provided when we study homology in a purely algebraic setting. At present, we merely wish to complete the topological tale [8].

For each fixed $n \geq 0$, we claim H, : Top $\to$ Ab is a functor. It remains to define, for every continuous f : X $\to$ Y and every $n \geq 0$, homomorphisms $H_n(f) : H_n(X) \to H_n(Y)$. This is done as follows. As a preliminary step, define "chain" homomorphisms $f_\#$: $S_n(X) \to S_n(Y)$ by $\sigma \mapsto f \circ \sigma$ (where σ is an n-simplex in X) and extend by linearity. A simple calculation shows the following diagram commutes :

$$\begin{tikzcd} S_n(X) \arrow[r, "\partial"] \arrow[d, "f_\#"'] & S_{n-1}(X) \arrow[d, "f_\#"] \\ S_n(Y) \arrow[r, "\partial"'] & S_{n-1}(Y) \end{tikzcd}$$

Figure (xiii)

i.e., $\partial f_\# = f_\#\,\partial$ (we have abused notation!). It follows easily that $f_\#\,(Z_n(X)) \subset Z_n(Y)$ and $f_\#\,(B_n(X)) \subset B_n(Y)$, so that $f_\#$ induces a well defined homomorphism between the quotients $H_n(X) \to H_n(Y)$ by

 ISSN: 2454-4248
53 – 60

$$H_n(f)\colon\ z_n + B_n(X) \mapsto f_\#(z_n) + B_n(Y),$$

where $z_n \in Z_n(X)$. Each H_n is, indeed, a functor.

Topological (and analytical) necessities require two modifications of this construction. If G is a fixed abelian group, replace the sequence

$$\ldots. \to S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \to \ldots\ldots.$$

by the sequence

$$\ldots.\ S_n(X) \otimes z\, G \xrightarrow{\partial_n \otimes 1_G} S_{n-1}(X) \otimes z\, G \to \ldots\ldots\ldots$$

By previous theorem we know that the composite of adjacent maps is 0 so that we may, as above, define cycles, boundaries, and homology. The groups so obtained are denoted $H_n(X ; G)$ and are called homology groups with coefficients G. In particular, since we know shows that our original construction yields the group $H_n(X ; Z)$.

The second modification constructs contravariant functors, called cohomology. If G is a fixed abelian group, replace the sequence

$$\ldots \to S_n(X) \xrightarrow{\partial_n} S_{n-1}(X) \to \ldots..$$

by the sequence of "cochains"

$$\ldots.. \leftarrow \mathrm{Hom}_z(S_n(X), G) \xrightarrow{\partial_n} \mathrm{Hom}_z(Sn_{-1}(X), G) \leftarrow\ldots\ldots$$

The arrows have changed direction because Hom_z (, G) is contravariant. Again, additive functors preserve zero morphisms, so the composite of adjacent maps is still 0. Certain subgroups of $\mathrm{Hom}_z(S_n(X), G)$ are defined, "cocycles" and "coboundaries", and their quotient H^h (X; G) is called the nth cohomology group of X with coefficients G. For each n ≥ 0, H^n(; G: Top → Ab is a contravariant functor. If one lets G = R, the additive group of reals, this is the correct context in which to simultaneously view the Fundamental Theorem of Calculus, Green's theorem. Stokes' theorem, and higher dimension analogues (de Rham theorem) [9-11].

We end this by exhibiting an algebraic context in which one constructs a long sequence of modules and maps in which the composite of adjacent maps is 0. Every module M can be described by generators and relations i.e., there is a map $F_0 \to M$ of a "free" module F_0 onto M with kernel K_0, say. Now K_0, in turn may also be so described : there is a map $F_1 \to K_0$ of a free module F_1 onto K_0 with kernel K_1, say. Link these together to get

$$\begin{tikzcd} K_1 \arrow[r] & F_1 \arrow[r] \arrow[dr] & F_0 \arrow[r] \arrow[d] & M \\ & & K_0 & \end{tikzcd}$$

where $F_1 \to F_0$ is defined as the composite $F_1 \to K_0 \to F_0$. This procedure may be iterated indefinitely to give a sequence

$$\ldots.. \to F_n \to F_{n-:} \to \ldots.. \to F_0 \to M \to 0,$$

where each F_n is free and composites of adjacent maps are 0. Both of the topologists modifications are available: for fixed module B, one may apply the function ⊗ B to obtain a new sequence and construct homology functions: one may apply Hom (B), to obtain a new sequence and construct contravariant cohomology functions.

III. Hom and ⊗

Homological algebra studies a ring R by investigating its category of modules $_R M$; this category, in turn, is investigated by examining the behavior of certain functions on it, the most important of which are Hom, ⊗, and related functions derived from these [12 & 13].

There are at least two reasons why this approach should be successful. The fancier reason is a theorem of Morita: two commutative rings R and S are isomorphic if and only if the categories $_R M$ and $_S M$ are "equivalent"; actually, Morita's theorem gives a necessary and sufficient condition on any pair of (not necessarily commulative) rings R and S that their module categories be equivalent. This theorem thus shows that the category $_R M$ conveys much information about R. Of course, there is a much more elementary way to see this. Recall that a left R-module M is an abelian group with a scalar multiplication $\sigma\colon R \times M \to M$. The module axioms assert that σ is Z-biadditive. Thus, for every fixed r ∈ R, the function $\sigma_r\colon M \to M$ defined by $m \mapsto \sigma(r, m) = rm$ is a Z-homomorphism. Now $\mathrm{End}_z(M) = \mathrm{Hom}_z(M, M)$ is a ring if we define multiplication as composition, and it is easy to see that $\rho : R \to \mathrm{End}_z(M)$ defined by $r \mapsto \sigma_r$ is a ring map. Thus, every R-module M defines a representatives of R in the endomorphism rings of an abelian group. Converesly, every such representation $\rho\colon R \to \mathrm{End}_z(M)$ makes the abelian group M into a left R-module by defining $\sigma\colon R \times M \to M$ by $(r,m) \mapsto \rho_r(m)$. Module theory is thus representation theory of rings.

Let us now look at module categories. Our initial observations essentially say that usual first properties of abelian groups and of vector spaces are also properties of more general modules.

Let R be a fixed ring (always associative with 1); we shall say "module" instead of "left R-module". Of course, all goes equally well for right modules, since we know that shows that every right R-module is a left R^{op}-module.

3.1 Definition : If M is a module, then a sub-module M' of M is a subgroup that is closed under scalar multiplication :

$$m' \in M' \text{ implies } rm' \in M', \quad \text{all } r \in R.$$

Examples 1. 0 and M are sub-modules of M; any sub-module M' ≠ M is called proper.

Examples 2. If M = R, its sub-modules are precisely the left ideals.

Examples 3. If I is a left ideal of R, then

$$IM = \{\Sigma a_j m_j : a_j \in I, m_j \in M\} \text{ is a sub-module of M.}$$

Examples 4. If I is a two-sided ideal of R (so that R/I is a ring) and if M is a module with IM = 0, then M is an R/I-module (if $\bar{r}$ = r +1, define $\bar{r}$ m, = rm).

Examples 5. Let f : M → N be an R-map. Then

$\ker f = \{m \in M: fm = 0\}$ is a sub-module of M, and

if $f = f(M) = \{n \in N: n = f(m) \text{ for some } m \in M\}$

is a sub-module of N. Of course, we have abbreviated the words kernel and image.

Examples 6. If M_1 and M_2 are sub-modules of M, then so is

$$M_1 + M_2 = \{m_1 + m_2 : m_1 \in M_1, m_2, \in M_2\}.$$

Examples 7. If $\{M'_j : j \in J\}$ is a family sub-modules of M, then $\cap_{j\in J} M'_j$ is also a sub-module of M.

3.2 Definition : Let X be a subset of a module M. The sub-module of M generated by X is $\cap_{j\in J} M'_j$, where $\{M'_j : j \in J\}$ is the family of all sub-modules of M that contain X. We denote this sub-module by $\langle X \rangle$.

3.3 Theorem : Let X be a subset of M. If X = ∅, then $\langle X \rangle$ = 0; if X ≠ ∅, then $\langle X \rangle = \{\Sigma r_i x_i : r_i \in R, x_i \in X\}$

Proof. : If X = ∅, then 0 is a sub-module of M containing X, from which it follows that $\langle \varnothing \rangle$ = 0. If X ≠ ∅, then the subset S = $\{\Sigma r_i x_i : r_i \in R, x_i \in X\}$ is defined (it is defined when X = ∅ if one enjoys summing over an empty index set). Since R contains 1, we have X ⊂ S. An easy check shows S is a sub-module of M, so it follows at once that $\langle X \rangle \subset S$. For the reverse inclusion, it suffices to show that if M' is any submodule of M containing X, then S ⊂ M' (for then S is contained in the intersection of all such M', which is $\langle X \rangle$). This is clear: $x_i \in M'$, all i, implies $\Sigma r_i x_i \in M'$ for all $r_i \in R$.

3.4 Definition: A module M is finitely generated (f.g.) if there is a finite subset $\{x_1, ..., x_n\}$ of M with $<x_1, ..., x_n> = M$; a module M is cyclic if there is a single element $x \in M$ with $\langle X \rangle$ = M.

3.5 Definition: Let f : M → N be an R-map. We say f is **monic** (or is a **monomorphism**) if f is one-one; we say f is epic (or is an **epimorphism**) if f is onto.

Of course, f is an **isomorphism** if and only if f is both monic and epic.

3.6 Definition: If M' is a sub-module of M, the **quotient module** M/M' is the quotient group M/M' made into an R-module by

$$r(m + M') = rm + M'$$

One must assume M' is a sub-module in order that the action of R on M/M' be well defined.

Examples 8. If M' is a sub-module of M, the inclusion i:M' → M is monic.

Examples 9. If M' is a sub-module of M, the natural map π :M → M/M' defined by m ⟼ m + M' is epic, and ker π = M'.

Examples 10. If f : M → N, then f is monic if and only if ker f = 0.

Examples 11. If f : M → N, then f is epic if and only if coker f = 0 (cokernel f is defined as the quotient module N/im f).

 ISSN: 2454-4248

Examples 12. **(First isomorphism Theorem)** If $f : M \to N$, then the map $m + \ker f \mapsto f(m)$ is an isomorphism $M/\ker f \xrightarrow{\cong} m f$.

Examples 13. **(Second Isomorphism Theorem)** If M_1 and M_2 are sub-modules of M, then $m_1 + M_1 \cap M_2 \mapsto m_1 + M_2$ is an isomorphism

$$M_1/M_1 \cap M_2 \xrightarrow{\cong} (M_1 + M_2)/M_2.$$

The second Isomorphism Theorem follows easily from the first : let $\pi : M \to M/M_1$ be the natural map, and let $f = \pi/M_1$. It is easy to see that ker $f = M_1 \cap M_2$ and im $f = (M_1 + M_2)/M_2$.

Examples 14. **(Third Isomorphism Theorem)** If $M2 \subset M1$ are sub-modules of M, then $(M/M_2) / (M_1/M_2) \cong M/M_1$ Third Isomorphism Theorem also follows easily from the First: the map $f : M/M_2 \to M/M_1$ given by $m + M_2 \mapsto m + M_1$ is epic with kernel M_1/M_2.

Examples 15. (Correspondence Theorem) If M' is a sub-module of M, there is a one-one correspondence between the sub-modules S of M/M' and the "intermediate" sub-modules of M containing M' given by $S \mapsto \pi^{-1}(S)$ (where $\pi : M \to M/M'$ is the natural map).

3.7 Theorem : A module M is cyclic if and only if $M \cong R/I$ for some left ideal I. Moreoever, if $M = \langle x \rangle$. then $I = \{r \in R : rx = 0\}$

Proof. First of all, R/I is cyclic with generator 1 + I; if $f : R/I \to M$ is an isomorphism, then $M = \langle x \rangle$, where $x = f(1+I)$. Conversely, assume $M = \langle x \rangle$. Define $f : R \to M$ by $f(r) = rx$. Since f is epic, $M \cong R/\ker f$. But ker f is a submodule of R, which is a left idea; indeed, $\ker f = \{r \in R : rx = 0\}$.

3.8 Definition : Two maps

$$M' \xrightarrow{f} M \xrightarrow{g} M''.$$

are exact at M if im f = ker g. A sequence of maps (perhaps infinitely long)

$$\ldots\ldots \to M_{n+1} \xrightarrow{fn-1} Mn \xrightarrow{fn} M_{n-1} \to \ldots\ldots$$

is exact if each adjacent pair of maps is exact.

Examples 16. If $0 \to M' \xrightarrow{f} M$ is exact, then f is monic (there is no need to label the only possible map $0 \to$ 'M'); if $M \xrightarrow{0} M'' \to 0$ is exact, then g is epic; if $0 \to M \xrightarrow{f} M' \to 0$ is exact, then f is an isomorphism.

Examples 17. If $M' \xrightarrow{f} M' \xrightarrow{g} M''$ is exact with f epic and g monic, then M = 0. Conclude that exactness of $0 \to M \to 0$ gives M = 0.

Examples 18. Prove that a map ϕ is monic iff $\varphi f = \varphi g$ implies $f = g$ (the diagram is $A \overset{f}{\underset{g}{\rightrightarrows}} B \xrightarrow{\phi} C$); prove that φ is epic if and only if $h\varphi = k\varphi$ implies $h = k$.

Examples 19. If $M_1 \xrightarrow{f} M_2 \to M_3 \xrightarrow{g} M_4$ is exact, then f is epic if and only if g is monic.

Examples 20. If $M_1 \xrightarrow{f} M_2 \to M_3 \to M_4 \xrightarrow{0} M_5$ is exact, then f epic and g monic imply $M_3 = 0$

Examples 21. If $0 \to M' \xrightarrow{i} M \to M'' \to 0$ is exact, then $M' \cong iM$ and $M/iM \cong M''$. Such sequences are called short exact sequences.

Examples 22. Consider the cummulative diagram with exact rows.

 ISSN: 2454-4248

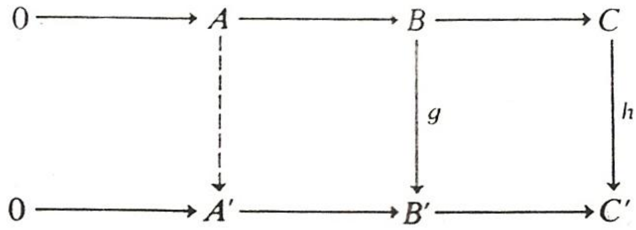

Figure (xix)

Prove that there exists a unique map $A \to A'$ making the diagram commute. Similarly, one can uniquely complete the commulative diagram with exact rows.

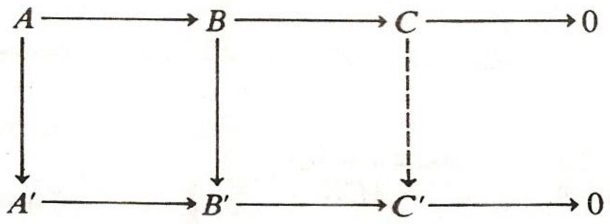

Figure (xx)

Remarks : There is a categorical translation of Example 2.7.Let U denote the category whose objects are all R-maps; define a morphism $\varphi : f \to g$ as a pair of maps $\varphi = (\varphi_1, \varphi_2)$ making the following diagram commute:

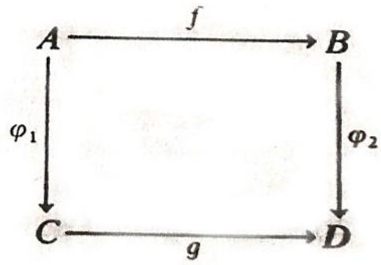

Figure (xxi)

One may now see that ker and coker are functors $U \to_R M$.

Example 23. If $f : M \to N$ is a map, there is an exact sequence

$$0 \to \ker f \to M \xrightarrow{f} N \to \operatorname{coker} f \to 0.$$

Examples 23. (Restatement of Third Isomorphism Theorem) If $M_2 \subset M_1$ are sub-modules of M, there is a short exact sequence $0 \to M_1/M_2 \to M/M_2\ M/M_1 \to 0$.

Examples 24. (Another Version of Third Isomorphism Theorem) Consider the commulative diagram.

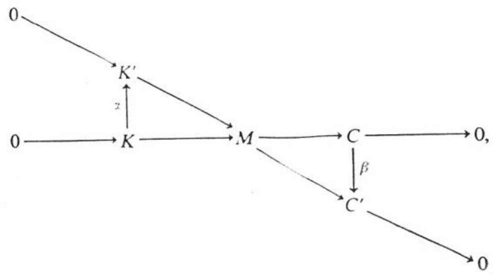

Figure (xxii)

where x is monic and β is epic. Then ker : $\neq 0$ if and only if coker $\alpha \neq 0$, i.e., $\alpha K'$ is a proper submodule of K. (Hint: Write $C = M/K$ and $C' = M/K'$, so that $\ker \beta = K'/K$)

IV. CONCLUSION

It is concluded that there are many definitions and theorems on homology theory which has been proved giving suitable examples. These examples verify the contribution of homology theory.

ISSN: 2454-4248
53 – 60

REFERENCES

[1]. V.S. Gubaa, M.V. Sapirb, Diagram groups and directed 2-complexes: Homotopy and homology. Journal of Pure and Applied Algebra 205 (2006) 1–47.
[2]. J. Brazas, Seme coverings : A generalization of covering space theory. Homology, Homotopy Appl. 14, 33-63, 2012.
[3]. J. Calcut, J. McCarthy, Discreteness and homogeneity of the topological fundamental group. Topology Proc. 34, 339-349, 2009.
[4]. Robert M Switzer. Algebraic topology-homotopy and homology, Springer, 2017. Books.google.com
[5]. R E Mosher, M C Tangora. Cohomology operations and applications in homotopy theory, 2008. Books.google.com
[6]. J.W. Vick, Homology theory: an introduction to algebraic topology, 2012. Books.google.com
[7]. A Pettet, The Johnson homomorphism and the second cohomology of IAn, Algebr. Geom. Topol. 5 (2005) 725–740 MR2153110
[8]. Craig Jensen, Jon McCammond, John Meier, The integral cohomology of the group of loops. Geometry and Topology, 10, 759-784, 2006.
[9]. Christel Rotthaus, Liana M. Sega, Some properties of graded local cohomology modules. Journal of Algebra, 283, 232-247, 2005.
[10]. J.M.,Berrio,R.,González-Díaz,F.Leal,M.M.López,P.Real,Visualizing cohomology aspects of 3D objects, Proceedings of the Sixth Asian Technical Conference in Mathematics, 2001, pp. 459–468.
[11]. R. González-Díaz, P. Real, Computation of cohomology operations on finite simplicial complexes, Homol. Homotopy Appl. 5 (2) (2003) 83–93.
[12]. R. González-Díaz, P. Real, Towards digital cohomology, DGCI 2003, Lecture Notes in Computer Science, vol. 2886, Springer, Berlin, 2003, pp. 92–101.
[13]. Rocío González-Díaz, Pedro Real, On the cohomology of 3D digital images Discrete Applied Mathematics 147 (2005) 245–263.