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# On Certain Ganeralized Fractional q-integral Operator of p-valent Funcions

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***Abstract***— In this paper, we consider generalized integral operator using fractional q-derivative together with Ruscheweyh derivative operator. And then use it to define new subclasses of p-valent functions in the open unit disk. Also we obtain two q-integral operators and study their sufficient conditions on subclasses of p-valent functions.

***Keywords-*** *fractional q-calculus operator, Ruscheweyh derivative q-integral operator, p-valent functions.*

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## I. INTRODUCTION

Let $A_p$ be the class of analytic and p-valent functions defined in the punchered open unit disk $U = \{z: |z| < 1\}$ is of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1,2,3,\dots\}), \tag{1}$$

A function $f \in A_p$ is said to be in the class p-valently starlike of complex order $b$ ($b \in \mathbb{C} - \{0\}$) and type $\gamma$ ( $0 \le \gamma \le p$), that is $f \in S_p(b,\gamma)$ if it satisfies the following condition

$$\Re\left\{p + \frac{1}{b}\left(\frac{z f'(z)}{f(z)} - p\right)\right\} > \gamma, \quad z \in U$$

On the other hand , A function $f \in A_p$ is said to be in the class p-valently convex of complex order $b$ ($b \in \mathbb{C} - \{0\}$) and type $\gamma$ ( $0 \le \gamma \le p$), that is $f \in K_p(b,\gamma)$ if it satisfies the following condition

$$\Re\left\{p + \frac{1}{b}\left(1 + \frac{z f''(z)}{f'(z)} - p\right)\right\} > \gamma, \quad z \in U.$$

In particular, for p=1, the classes $S_1(b,\gamma)$ and $K_1(b,\gamma)$ are respectively the classes $S(b,\gamma)$ and $K(b,\gamma)$ that were introduced by Frasin [1].

In the theory of q-calculus, the q-shifted factorial is defined for $\alpha, q \in \mathbb{C}, n \in \mathbb{N}_0 \equiv \mathbb{N} \cup \{0\}$ as a product of n factors by

$$(\alpha; q)_n = \begin{cases} 1 & , n = 0 \\ (1-\alpha)(1-\alpha q)\dots\dots(1-\alpha q^{n-1}), & n \in \mathbb{N}, \end{cases} \tag{2}$$

and in terms ofthe basic analogue of the Gamma function

$$(q^\alpha ; q)_n = \frac{\Gamma_q(\alpha+n)(1-q)^n}{\Gamma_q(\alpha)}, \quad (n > 0),$$

Where the q-gamma function [ 2,3] is defined by

$$\Gamma_q(x) = \frac{(q;q)_\infty (1-q)^{1-x}}{(q^x;q)_\infty}, \quad (0 < q < 1).$$

If $|q| < 1$, the q-shifted factorial (2) remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha ; q)_\infty = \prod_{j=0}^{\infty}(1-\alpha q^j).$$

It is well known that $\Gamma_q(x) \to \Gamma(x)$ as $q \to 1$, where $\Gamma(x)$ is the ordinary Eular gamma function.

In view of the relation

$$\lim_{q \to 1^-} \frac{(q^\alpha ; q)_n}{(1-q)^n} = (\alpha)_n,$$

we observe that the q-shifted factorial (2) reduces to the familiar Pochhammer symbol $(\alpha)_n$, where $(\alpha)_n = \alpha(\alpha-1)..(\alpha+n-1)$ .

the Jackson's q-derivative and q-integral of a function on subset of $\mathbb{C}$ are, respectively, given by Gasper and Rahman [2, pp 19-22.]

$$D_{q,z} f(z) = \frac{f(z) - f(zq)}{z(1-q)}, (z \neq 0, q \neq 0) \tag{3}$$

and

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$$\int_0^z f(t)\, d_q(t) = z(1-q)\sum_{k=0}^{\infty} q^k f(zq^k). \tag{4}$$

Therefore, q-derivative of $f(z) = z^n$, where n is a positive integer is given by

$$D_q z^n = \frac{z^n - (zq)^n}{(1-q)z} = [n]_q\, z^{n-1},$$

where

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1},$$

And is called q-analogue of n, as q → 1, we have $[n]_q = 1 + q + q^2 + \cdots + q^{n-1} \to 1 + 1 + \cdots + 1 = n$.

We now recall the definitions of fractional q-calculus operators of a complex-valued function f(z), which were recently studied by Purohit and Raina [4].

**Definition 1.** (Fractional q-integral operator) Let f(z) is analytic in a simply connected regon of the z-plane containing the origin. The fractional q-integral of f of order $\delta$ $(\delta > 0)$ is defined by

$$I_{q,z}^{\delta} f(z) = D_{q,z}^{-\delta} f(z) = \frac{1}{\Gamma_q(\delta)} \int_0^z (z-qt)_{\delta-1} f(t)\, d_q t, \tag{5}$$

Where $(z-tq)_{\delta-1}$ can be expressed as q- bionomial given by

$$(z-tq)_{\delta-1} = z^{\delta-1}\phi_0\,[q^{-\delta+1}; -; q; \tfrac{tq^{\delta}}{z}] \tag{6}$$

The series $\phi_0\,[\delta; -; q; z]$ is a single valued when $|\arg(z)| < \pi$, $|z| < 1$, therefore the function $(z-tq)_{\delta-1}$ in (6) is single valued when $\left|arg\left(\frac{-tq^{\delta}}{z}\right)\right| < \pi$, $|tq^{\delta}| < 1$ and $||arg(z)| < \pi$.

**Definition 2 .** (Fractional q-derivative operator) The fractional q- derivative operator of order $\delta$ $(0 \le \delta < 1)$ is defined by function f(z) by

$$D_{q,z}^{\delta} f(z) = D_{q,z}\, I_{q,z}^{1-\delta} f(z) = \frac{1}{\Gamma_q(1-\delta)} D_{q,z} \int_0^z (z-qt)_{-\delta} f(t) d_q t \tag{7}$$

Where the function f(z) is constrained , and the multiplicity of function $(z-qt)_{-\delta}$ is removed as in Definition 1.

**Definition 3.** (Extended fractional q-derivative operator) Under the hypothesis of Definition 2 , the fractional q-derivative of order $\delta$ is defined by

$$D_{q,z}^{\delta} f(z) = D_{q,z}^{m}\, I_{q,z}^{m-\delta} f(z),\ (m-1 \le \delta < m\,; m \in \mathbb{N}_0). \tag{8}$$

By using known extensions involving q-differ-integral operator ,we define the Linear opartor

$$\Omega_{q,p}^{\delta} f(z) : \mathrm{I} \to \mathrm{I}$$

$$\Omega_{q,p}^{\delta} f(z) = \frac{\Gamma_q(p+1-\delta)}{\Gamma_q(p+1)} z^{\delta} D_{q,z}^{\delta} f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{\Gamma_q(p+1-\delta)\Gamma_q(n+1)}{\Gamma_q(p+1)\Gamma_q(n+1-\delta)} a_n z^n \tag{9}$$

Where we can easily check that $D_{q,z}^{\delta} f(z)$ in (9) represents, respectively, a fractional q-integral of f(z) of order δ when $-\infty < \delta < 0$ and a fractional q-derivative of f(z) of order δ when $-\infty < \delta < p+1$ . By (9), it is easy to see $\Omega_{q,p}^{0} f(z) = f(z)$

Next we recall the generalized AL-Oboudi type differential operator $D_{q,p,\lambda}^{\delta,m} : A_p \to A_p, for\ \lambda \ge 0\ and\ m \in \mathbb{N}$, as follow:

$$D_{q,p,\lambda}^{\delta,0} f(z) = f(z)$$

$$D_{q,p,\lambda}^{\delta,1} f(z) = (1-\lambda)\,\Omega_{q,p}^{\delta} f(z) + \frac{\lambda z}{[p]_q} D_q \left(\Omega_{q,p}^{\delta} f(z)\right),$$

$$D_{q,p,\lambda}^{\delta,2} f(z) = D_{q,p,\lambda}^{\delta,1} f(z) \left(D_{q,p,\lambda}^{\delta,1} f(z)\right),$$

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$$D_{q,p,\lambda}^{\delta,m} f(z) = D_{q,p,\lambda}^{\delta,1} f(z) \left(D_{q,p,\lambda}^{\delta,m-1} f(z)\right), \tag{10}$$

Which was introduce by Selvakumaran et al. [5]. We note that, if $f \in A_p$ is given by (1), then by (10) we have

$$D_{q,p,\lambda}^{\delta,m} f(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{\Gamma_q(p+1-\delta)\Gamma_q(n+1)}{\Gamma_q(p+1)\Gamma_q(n+1-\delta)} \left[1 - \lambda + \frac{[n]_q}{[p]_q}\lambda\right]\right)^m a_n z^n.$$

We note that, by specializing the parameters in the operator $D_{q,p,\lambda}^{\delta,m}$, this operator reduces to many well known differential operators. For example when p=1, and q , this operator reduces to many well known differential operators. For example when p=1, and q → 1, the operator $D_{q,p,\lambda}^{\delta,m}$ reduces to the operator introduced by AL-Oboudi and AL-Amoudi [6]. Also if δ =0, p=1, and q → 1, it become the operator $D_{\lambda}^{n}$ introduced by AL-Oboudi [7]. The special case $D_0^n = D^n$ was consider by Salagean [8].

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Analogously, we define the generalized Ruschewyh type differential operator $R_{q,p,\lambda}^{\delta,m} : A_p \to A_p, for\ m \in \mathbb{N}$, as follows:

$$R_{q,p,\lambda}^{\delta,0} f(z) = f(z),$$
$$R_{q,p,\lambda}^{\delta,1} f(z) = \frac{z}{[p]_q} D_{q,p} \left(\Omega_{q,p}^{\delta} f(z)\right),$$
$$2\, R_{q,p,\lambda}^{\delta,2} f(z) = \frac{z}{[p]_q} D_q \left(R_{q,p,\lambda}^{\delta,1} f(z)\right) + R_{q,p,\lambda}^{\delta,1} f(z),$$
$$\vdots$$
$$(m+1) R_{q,p,\lambda}^{\delta,m+1} f(z) = \frac{z}{[p]_q} D_q \left(R_{q,p,\lambda}^{\delta,m} f(z)\right) + m\, R_{q,p,\lambda}^{\delta,m} f(z), \tag{11}$$

For $f \in A_p$ is given by (1), then by (11) we have

$$R_{q,p,\lambda}^{\delta,m} f(z) = z^p + \sum_{n=p+1}^{\infty} \left(\frac{\Gamma_q(p+1-\delta)\Gamma_q(n+1)}{\Gamma_q(p+1)\Gamma_q(n+1-\delta)}\right)^m C_m\left(\frac{[n]_q}{[p]_q}\right) a_n\, z^n. \tag{12}$$

where

$$C_m(t) = \frac{t(t+1)\ldots\ldots(t+m),}{m!}$$

We note that, for δ =0, p=1, and q → 0, the operator $R_{q,1}^{0,m}$ becomes the operator introduced by Ruscheweyh [9].

The q-analogues to the function classes $S_p(b,\gamma)$ and $K_p(b,\gamma)$ are given as follows:

A function $f \in A_p$ is said to be in the class $S_{q,p}(b,\gamma)$ p-valently starlike of complex order $b$ ($b \in \mathbb{C} - \{0\}$) and type $\gamma$ ( $0 \le \gamma \le p$), with respect to q-differentiation if it satisfies the following condition

$$\Re\left\{[p]_q + \frac{1}{b}\left(\frac{z f'(z)}{f(z)} - [p]_q\right)\right\} > \gamma, \quad z \in U$$

On the other hand , A function $f \in A_p$ is said to be in the class $K_{q,p}(b,\gamma)$ p-valently convex of complex order $b$ ($b \in \mathbb{C} - \{0\}$) and type $\gamma$ ( $0 \le \gamma \le p$), with respect to q-differentiation if it satisfies the following condition

$$\Re\left\{[p]_q + \frac{1}{b}\left(1 + \frac{z f''(z)}{f'(z)} - [p]_q\right)\right\} > \gamma, \quad z \in U.$$

By using the operator $R_{q,p,\lambda}^{\delta,m} f(z)$ defined by (12) and q-differentiation, we introduce two new subclasses of analytic functions $S_{q,p,\lambda}^{\delta,m}(b,\ \gamma)$ and $K_{q,p,\lambda}^{\delta,m}(b,\ \gamma)$ as follows:

A function $f \in A_p$ is said to be p-valently starlike of complex order $b$ ($b \in \mathbb{C} - \{0\}$) and type $\gamma$ ( $0 \le \gamma \le p$), with respect to q-differentiation if it satisfies the following condition

$$\Re\left\{[p]_q + \frac{1}{b}\left(1 + \frac{z\, D_q\left(R_{q,p,\lambda}^{\delta,m} f(z)\right)}{R_{q,p,\lambda}^{\delta,m} f(z)} - [p]_q\right)\right\} > \gamma, \quad (\alpha \in [0,p],\ b \in \mathbb{C} - \{0\}, z \in U) \tag{13}$$

We say that f is in the class $S_{q,p,\lambda}^{\delta,m}(b,\ \gamma)$ for such functions. Denote by $S_q(p,\ b,\ \gamma)$, the subclass of $S_{q,p,\lambda}^{\delta,m}(b,\ \gamma)$ consisting of function $f \in A_p$ for which

$$\left|\frac{1}{b}\left(1 + \frac{z\, D_q\left(R_{q,p,\lambda}^{\delta,m} f(z)\right)}{R_{q,p,\lambda}^{\delta,m} f(z)} - [p]_q\right)\right| < [p]_q - \gamma, \quad z \in U$$

On the other hand, a function $f \in A_p$ is said to be p-valently convex of complex order $b$ ($b \in \mathbb{C} - \{0\}$) and type $\gamma$ ( $0 \le \gamma \le p$), with respect to q-differentiation if it satisfies the following condition

$$\Re\left\{[p]_q + \frac{1}{b}\left(\frac{z D_q^2\left(R_{q,p,\lambda}^{\delta,m} f(z)\right)}{D_q\left(R_{q,p,\lambda}^{\delta,m} f(z)\right)} - [p]_q\right)\right\} > \gamma, \quad (\alpha \in [0,p],\ b \in \mathbb{C} - \{0\}, z \in U). \tag{14}$$

Analogously, we denote the subclass $K_q(p,\ b,\ \gamma)$ of $K_{q,p,\lambda}^{\delta,m}(b,\ \gamma)$ consisting of functions $f \in A_p$ for which

$$\left|\frac{1}{b}\left(\frac{z D_q^2\left(R_{q,p,\lambda}^{\delta,m} f(z)\right)}{D_q\left(R_{q,p,\lambda}^{\delta,m} f(z)\right)} - [p]_q\right)\right| < [p]_q - \gamma, \quad z \in U$$

We now consider the p-valent q-integral operator using the operator $R_{q,p,\lambda}^{\delta,m} f(z)$ defined by (12) as follows:

**Definition 4:** Let $m_i \in \mathbb{N}_0$ , $\mu_i \in \mathbb{R}^+$ *and* $f_i \in A_p$ *for all* $i = 1,2,\ldots,n$. Then $F_{q,n} : A_p^n \to A_p$ is defined as

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$$F_{q,n}(z) = \mathscr{F}_{q,p,\lambda}^{\delta,\gamma,m}(f_1, f_2, \ldots\ldots, f_n)(z) = \int_0^z [p]_q\, t^{p-1} \prod_{i=1}^{n} \left(\frac{R_{q,p,\lambda}^{\delta,m_i} f_i(t)}{t^p}\right)^{\gamma_i} d_q t \tag{15}$$

And $G_{q,n} : A_p^n \to A_p$ is defined as

$$G_{q,n}(z) = \mathcal{G}_{q,p,\lambda}^{\delta,\gamma,m}(f_1, f_2, \ldots\ldots, f_n)(z) = \int_0^z [p]_q\, t^{p-1} \prod_{i=1}^{n} \left(\frac{D_q\left(R_{q,p,\lambda}^{\delta,m_i} f_i(t)\right)}{[p]_q\, t^{p-1}}\right)^{\gamma_i} d_q t \tag{16}$$

In the next section, we study some properties of the q-integral operators $F_{q,n}(z)$ and $G_{q,n}(z)$ defined by (15) and (16) respectively, on the classes $S_q(p, b, \gamma)$ and $K_q(p, b, \gamma)$. As special cases, the order of convexity of the operators $\int_0^z \left(\frac{f(t)}{t}\right)^{\gamma} dt$ and $\int_0^z \left(f'(t)\right)^{\gamma} dt$ are also given.

## II. SUFFICIENT CONDITIONS FOR THE OPERATOR $F_{q,n}(z)$

**Theorem1**. Let $m_i \in \mathbb{N}_0, \gamma_i \in \mathbb{R}^+, 0 \le \alpha_i \le p,\ b \in \mathbb{C} - \{0\}$ *and* $R_{q,p,\lambda}^{\delta,m_i} f_i(z) \in S_q(p, b, \gamma_i)$ *for all* $i = 1,2,\ldots,n$, $n \in \mathbb{N}$. If

$$0 \le [p]_q + \frac{(p - [p]_q)}{|b|}\left|1 - \sum_{i=1}^{n} \gamma_i\right| + \sum_{i=1}^{n} \gamma_i([p]_q - \gamma_i) < p$$

Then the q-integral operator $F_{q,n}(z)$ defined by (15) is in the class $S_q(p, b, \zeta)$, where

$$\zeta = [p]_q + \frac{(p - [p]_q)}{|b|}\left|1 - \sum_{i=1}^{n} \gamma_i\right| + \sum_{i=1}^{n} \gamma_i([p]_q - \gamma_i)$$

**Proof:** From (15), we observe that $F_{q,n}(z) \in A_p$. On the other hand, it is easy to verify that

$$D_q\left(F_{q,n}(z)\right) = [p]_q\, z^{p-1} \prod_{i=1}^{n} \left(\frac{R_{q,p,\lambda}^{\delta,m_i} f_i(z)}{z^p}\right)^{\gamma_i}.$$

Now by logarithmic q-differentiation we have,

$$\frac{\ln q}{q-1}\left[\frac{D_q^2\left(F_{q,n} f(z)\right)}{D_q\left(F_{q,n} f(z)\right)}\right] = \frac{\ln q}{q-1}\left[\frac{p-1}{z} + \sum_{i=1}^{n} \gamma_i \left(\frac{D_q\left(R_{q,p,\lambda}^{\delta,m_i} f_i(z)\right)}{R_{q,p,\lambda}^{\delta,m_i} f_i(z)} - \frac{p}{z}\right)\right].$$

Therefore,

$$\frac{z\, D_q^2\left(F_{q,n} f(z)\right)}{D_q\left(F_{q,n} f(z)\right)} = p - 1 + \sum_{i=1}^{n} \gamma_i \left(\frac{z\, D_q\left(R_{q,p,\lambda}^{\delta,m_i} f_i(z)\right)}{R_{q,p,\lambda}^{\delta,m_i} f_i(z)} - p\right).$$

Therefore, by multiplying z and some calculations give

$$\frac{z\, D_q^2\left(F_{q,n} f(z)\right)}{D_q\left(F_{q,n} f(z)\right)} - [p]_q = \sum_{i=1}^{n} \gamma_i \left(\frac{z\, D_q\left(R_{q,p,\lambda}^{\delta,m_i} f_i(z)\right)}{R_{q,p,\lambda}^{\delta,m_i} f_i(z)} - [p]_q\right) + (p - [p]_q)\left(1 - \sum_{i=1}^{n} \gamma_i\right).$$

Then by multiplying above equation by 1/b , we obtain that

$$\left|\frac{1}{b}\left(1 + \frac{z\, D_q^2\left(F_{q,n} f(z)\right)}{D_q\left(F_{q,n} f(z)\right)} - [p]_q\right)\right| \le \frac{(p - [p]_q)}{|b|}\left|1 - \sum_{i=1}^{n} \gamma_i\right| + \sum_{i=1}^{n} \gamma_i \left|\frac{1}{b}\left(\frac{z\, D_q\left(R_{q,p,\lambda}^{\delta,m_i} f_i(z)\right)}{R_{q,p,\lambda}^{\delta,m_i} f_i(z)} - [p]_q\right)\right|.$$

Since $R_{q,p,\lambda}^{\delta,m_i} f_i(z) \in S_q(p, b, \alpha_i)$, above equation can be written as

$$\left|\frac{1}{b}\left(1 + \frac{z\, D_q^2\left(F_{q,n} f(z)\right)}{D_q\left(F_{q,n} f(z)\right)} - [p]_q\right)\right| \le \frac{(p - [p]_q)}{|b|}\left|1 - \sum_{i=1}^{n} \gamma_i\right| + \sum_{i=1}^{n} \gamma_i(\alpha_i - [p]_q).$$

Therefore, $F_{q,n}(z) \in S_q(p, b, \zeta)$. *N*ow, the proof is completed.

Taking p=1, b=1 and $m_i = 0$ for all I = 1,2,…..,n in Theorem 1 we have the following corollary.

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**Corollary 1:** Let $\gamma_i \in \mathbb{R}^+, 0 \leq \alpha_i \leq 1$ *and* $R_{q,p,\lambda}^{\delta,m_i} f_i(z) \in S_q(\gamma_i)$ *for all* $i = 1,2,\ldots,n,\ n \in \mathbb{N}$. If

$$0 \leq 1 + \sum_{i=1}^{n} \gamma_i(\alpha_i - 1) < 1$$

Then the q-integral operator

$$F_q(z) = \int_0^z \left(\frac{f_1(t)}{t}\right)^{\gamma_1} \ldots \left(\frac{f_n(t)}{t}\right)^{\gamma_n} d_q t$$

In convex with respect to q-differentiation of order $\zeta$, where $\zeta = 1 + \sum_{i=1}^{n} \gamma_i(\alpha_i - 1)$.

**Theorem 2:** Let $m_i \in \mathbb{N}_0, \gamma_i \in \mathbb{R}^+, 0 \leq \alpha_i \leq p,\ b \in \mathbb{C} - \{0\}$ *and* $R_{q,p,\lambda}^{\delta,m_i} f_i(z) \in K_q(p, b, \gamma_i)$ *for all* $i = 1,2,\ldots,n$, $n \in \mathbb{N}$. If

$$0 \leq [p]_q + \frac{(p - [p]_q - 1)}{|b|}\left|1 - \sum_{i=1}^{n} \gamma_i\right| + \sum_{i=1}^{n} \gamma_i([p]_q - \alpha_i) < p$$

Then the q-integral operator $G_{q,n}(z)$ defined by (15) is in the class $K_q(p, b, \zeta)$, where

$$\zeta = [p]_q + \frac{(p - [p]_q - 1)}{|b|}\left|1 - \sum_{i=1}^{n} \gamma_i\right| + \sum_{i=1}^{n} \gamma_i([p]_q - \alpha_i)$$

Proof: From (15), we observe that $G_{q,n}(z) \in A_p$. On the other hand, it is easy to verify that

$$D_q\left(G_{q,n}(z)\right) = [p]_q z^{p-1} \prod_{i=1}^{n} \left(\frac{D_q\left(R_{q,p,\lambda}^{\delta,m_i} f_i(z)\right)}{[p]_q z^{p-1}}\right)^{\gamma_i}.$$

Now by logarithmic q-differentiation we have,

$$\frac{\ln q}{q-1}\left[\frac{D_q^2\left(G_{q,n} f(z)\right)}{D_q\left(G_{q,n} f(z)\right)}\right] = \frac{\ln q}{q-1}\left[\frac{p-1}{z} + \sum_{i=1}^{n} \gamma_i \left(\frac{D_q^2\left(R_{q,p,\lambda}^{\delta,m_i} f_i(z)\right)}{D_q\left(R_{q,p,\lambda}^{\delta,m_i} f_i(z)\right)} - \frac{p-1}{z}\right)\right].$$

Therefore,

$$\frac{z\, D_q^2\left(G_{q,n} f(z)\right)}{D_q\left(G_{q,n} f(z)\right)} = p - 1 + \sum_{i=1}^{n} \gamma_i \left(\frac{z\, D_q^2\left(R_{q,p,\lambda}^{\delta,m_i} f_i(z)\right)}{D_q\left(R_{q,p,\lambda}^{\delta,m_i} f_i(z)\right)} - (p-1)\right).$$

Therefore, by multiplying z and some calculations give

$$\frac{z\, D_q^2\left(G_{q,n} f(z)\right)}{D_q\left(G_{q,n} f(z)\right)} - [p]_q = \sum_{i=1}^{n} \gamma_i \left(\frac{z\, D_q^2\left(R_{q,p,\lambda}^{\delta,m_i} f_i(z)\right)}{D_q\left(R_{q,p,\lambda}^{\delta,m_i} f_i(z)\right)} - [p]_q\right) + (p - [p]_q - 1)\left(1 - \sum_{i=1}^{n} \gamma_i\right).$$

Then by multiplying above equation by 1/b , we obtain that

$$\left|\frac{1}{b}\left(\frac{z\, D_q^2\left(G_{q,n} f(z)\right)}{D_q\left(G_{q,n} f(z)\right)} - [p]_q\right)\right| \leq \frac{(p - [p]_q - 1)}{|b|}\left|1 - \sum_{i=1}^{n} \gamma_i\right| + \sum_{i=1}^{n} \gamma_i \left|\frac{1}{b}\left(\frac{z\, D_q\left(R_{q,p,\lambda}^{\delta,m_i} f_i(z)\right)}{R_{q,p,\lambda}^{\delta,m_i} f_i(z)} - [p]_q\right)\right|.$$

Since $R_{q,p,\lambda}^{\delta,m_i} f_i(z) \in K_q(p, b, \alpha_i)$, above equation can be written as

$$\left|\frac{1}{b}\left(\frac{z\, D_q^2\left(G_{q,n} f(z)\right)}{D_q\left(G_{q,n} f(z)\right)} - [p]_q\right)\right| \leq \frac{(p - [p]_q - 1)}{|b|}\left|1 - \sum_{i=1}^{n} \gamma_i\right| + \sum_{i=1}^{n} \gamma_i([p]_q - \alpha_i).$$

Therefore, $G_{q,n}(z) \in K_q(p, b, \zeta)$. *Now*, the proof is completed.

Taking p=1, b=1 and $m_i = 0$ for all I = 1,2,…..,n in Theorem 1 we have the following corollary.

**Corollary 2:** Let $\gamma_i \in \mathbb{R}^+, 0 \leq \alpha_i \leq 1$ *and* $R_{q,p,\lambda}^{\delta,m_i} f_i(z) \in K_q(\gamma_i)$ *for all* $i = 1,2,\ldots,n,\ n \in \mathbb{N}$. If

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$$0 \leq \sum_{i=1}^{n} \gamma_i (2 - \alpha_i) < 1$$

Then the q-integral operator

$$F_q(z) = \int_0^z \left(D_q f_1(t)\right)^{\gamma_1} \ldots . \left(D_q f_n(t)\right)^{\gamma_n} d_q t$$

In convex with respect to q-differentiation of order $\zeta$, where $\zeta = \sum_{i=1}^{n} \gamma_i (2 - \alpha_i)$.

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