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Unique Common Fixed Point Theorem for Non-Expansive Mappings

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Abstract— In the present paper two theorems on non-expansive mappings have been established. The first theorem is for four mappings which satisfy *R*- sub weakly commuting property in pair. The result of first theorem is used to develop another theorem for q-star shaped subset of a normed space. An attempt has been made to prove one more theorem for two weakly compatible mappings on convex sets. These results are the extension and generalization of earlier results existing in the literature.

Keywords- *Non-Expansive Mapping, R -weakly commuting, R - sub weakly commuting, q-star shaped set , convex set ,weakly compatible.*

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I. Introduction

Let (X,d) be a metric space, T a self-mapping on X and k a nonnegative real number such that the inequality $d(Tx,Ty)\le kd(x,y)$ holds for any $x,y\in X$. If $k<1$ then T is said to be a contractive mapping, if $k=1$, then T is said to be a non-expansive mapping. The well-known Banach theorem states that if X is complete then every contractive mapping has a unique fixed point, however, a non-expansive mapping need not have fixed points.

Huge research work has been carried on Banach contraction principle with different types of contraction conditions but significant research is not found in the direction of non-expansive mappings.

Bogin[2] proved the following result for a non-expansive mapping in a metric space

Theorem [2]: Let (X,d) be a nonempty complete metric space and $T:X\to X$ a mapping satisfying

$$d(Tx,Ty)\le ad(x,y)+b\big(d(x,Tx)+d(y,Ty)\big)$$
$$+c\big(d(x,Ty)+d(y,Tx)\big)$$

where $a\ge 0,b>0,c>0$ and $a+2b+2c=1$. Then T has a unique fixed point.

Ćirić [3] used a more generalized contractive condition and thus modified the above result.

Gregus [4] considered non-expansive mapping for convex set of a Banach space.

Shezad [5] proved the following theorem for two $R-$ weakly commuting mappings.

Theorem [5]: Let M be a closed subset of a metric space (X,d) and A and S be $R-$ weakly commuting self-mappings of X such that $A(M)\subseteq S(M)$. Suppose there exists $k\in(0,1]$ such that

$$d(Ax,Ay)\le k\max\left\{\begin{array}{l} d(Sx,Sy),d(Sx,Ax),d(Sy,Ay),\\ \frac{1}{2}\big(d(Sx,Ay),d(Sy,Ax)\big)\end{array}\right\}$$

for all $x,y\in M$. If $cls(S(M))$ and S is continuous, then $M\cap F(S)\cap F(A)$ is singleton.

Jungk et.al [1] have worked in different direction on non-expansive mappings for normed space and established the following result.

Theorem [1]: Let M be a nonempty q-star-shaped subset of a normed space X and A,S and T be self-maps of M . Suppose that S and T are linear and continuous with $q\in F(S)\cap F(T)$ and $A(M)\subset S(M)\cap T(M)$. If the pairs $\{A,S\}$ and $\{A,T\}$ are $R-$ subweakly commuting and satisfy,

$$\|Ax-Ay\|$$
$$\le\max\left\{\begin{array}{l}\|Sx-Ty\|,dist\big(Sx,[Ax,q]\big),dist\big(Ty,[Ay,q]\big),\\ \frac{1}{2}\Big(dist\big(Sx,[Ay,q]\big)+dist\big(Ty,[Ax,q]\big)\Big)\end{array}\right\}$$

for all $x,y\in M$,

then $F(A)\bigcap F(S)\bigcap F(T)\ne\phi$ provided one of the following conditions holds:

(i) M is complete, $cls(A(M))$ is compact and A is continuous,

(ii) M is weakly compact, $(S-A)$ is demiclosed at *0* and X is complete.

(iii) M is weakly compact and X is complete space satisfying Opial's condition.

In the present paper we have proved a theorem for four non expansive mappings which is a generalisation of Shehzad [5]. Using the first result the second theorem on q-star shaped subset of a normed space has been established. This theorem is a generalisation of Jungk et.al.[1] which is for three mappings.

The last theorem is on two weakly compatible mappings in metric space which is a modification as well as generalisation of Bogin [2].

II. Preliminary

Some basic definitions are necessary to discuss before we start the main theorems.

Let M be a nonempty subset of a normed space $(X,\|.\|)$, and let A,B and T be self-mappings of M

2.1- (A,B) -contraction: A mapping T is said to be (A,B) -contraction if there exists $k\in(0,1)$ such that

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$\|Tx - Ty\| \le k\|Ax - By\|$ for all $x, y \in M$. If $k = 1$, then T is said to be (A, B)-non-expansive. If $A = B$ then T is said to be A-contraction. If $k = 1$, then T is said to be A-non-expansive. If $A = I$ and $k = 1$, then T is said to be non-expansive. The set of fixed points of T (respectively A) is denoted by ($F(T)$) (respectively $F(A)$)

2.2- R-weakly commuting [6]: A mapping A and T are said to be R-weakly commuting on M, if there exists a real number $R > 0$ such that $\|TAx - ATx\| \le R\|Tx - Ax\|$ for all $x \in M$

2.3- q – star shaped: A set M is called q-star shaped with $q \in M$ if the segment $[q, x] = \{(1-k)q + kx : 0 \le k \le 1\}$ is contained in M for all $x \in M$.

2.4- R - sub weakly commuting [5]: If M is q-star shaped with $q \in F(A)$, then T and A are said to be R- sub weakly commuting on M if $\|TAx - ATx\| \le R\,dist(Ax, [Tx, q])$ for all $x \in M$ and $R > 0$, where $dist(Ax, [Tx, q]) = \inf\{\|Ax - y\| : y \in [Tx, q]\}$

It is well known that R-sub weakly commuting maps are R-weakly commuting and R-weakly commuting maps are compatible but not conversely.

2.5- Opial's condition [7]: A Banach space X satisfies Opial's condition if for every sequence $\{x_n\}$ in X is weakly convergent to $x \in X$, the inequality $\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|$ holds for all $y \ne x$. Every Hilbert space and the space $l_p\ (1 \le p < \infty)$ satisfy Opial's condition.

2.6- Demiclosed: The map $T : M \to X$ is said to be demiclosed at 0 if for every sequence $\{x_n\}$ in M such that $\{x_n\}$ converges weakly to x and $\{Tx_n\}$ converges strongly to $0 \in X$, then $0 = Tx$.

III. THEOREMS

We established the following theorem for four mappings which in pair are R-weakly commutating, is a generalisation of Shehzad [5]

3.1.1-Theorem: Let A, B, S, T be self-maps of a complete metric space (X, d). Suppose that S, T are continuous, the pairs (A, S) and (B, T) are R-weakly commutating and $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$. If there exists $\lambda \in [0,1)$ such that

$$d(Ax, By) \le \lambda\ M(x, y)$$

where

$$M(x,y) = \max\left\{\begin{array}{l} d(Sx,Ty), d(Ax,Sx), d(By,Ty), \\ \frac{1}{2}\left(d(Ax,Ty) + d(Sx,By)\right) \end{array}\right\} \quad \ldots.(1)$$

for $x, y \in X$. Then there is a unique fixed point $u \in X$ such that $Au = Bu = Su = Tu = u$.

Proof: Let $x_0 \in X$ and $x_1 \in X$ such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$ and $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$.

Putting $x = x_{2n}$ and $y = x_{2n+1}$ in (1), we have

$$d(Ax_{2n}, Bx_{2n+1}) \le \lambda\, M(x_{2n}, x_{2n+1})$$

$$M(x_{2n}, x_{2n+1}) = \max\left\{\begin{array}{l} d(Sx_{2n}, Tx_{2n+1}), d(Ax_{2n}, Sx_{2n}), \\ d(Bx_{2n+1}, Tx_{2n+1}), \\ \frac{1}{2}\left(d(Ax_{2n}, Tx_{2n+1}) + d(Sx_{2n}, Bx_{2n+1})\right) \end{array}\right\}$$

$$= \max\left\{\begin{array}{l} d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n-1}), d(y_{2n+1}, y_{2n}), \\ \frac{1}{2}\left(d(y_{2n}, y_{2n}) + d(y_{2n-1}, y_{2n+1})\right) \end{array}\right\}$$

$$= \max\left\{d(y_{2n-1}, y_{2n}), d(y_{2n+1}, y_{2n}), \frac{1}{2}d(y_{2n-1}, y_{2n+1})\right\} (2)$$

Since,

$$\frac{1}{2}d(y_{2n-1}, y_{2n+1}) \le \frac{1}{2}\left(d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})\right)$$
$$\le \max\left\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\right\}$$

$$\therefore d(Ax_{2n}, Bx_{2n+1}) \le \lambda \max\left\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\right\}$$

Case-I: If

$$\max\left\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\right\} = d(y_{2n}, y_{2n+1})$$

$$\therefore d(y_{2n}, y_{2n+1}) \le \lambda d(y_{2n}, y_{2n+1})$$

$\because \lambda \in (0,1)$. So, it is a contradiction.

Case-II: If

$$\max\left\{d(y_{2n-1}, y_{2n}), d(y_{2n}, y_{2n+1})\right\} = d(y_{2n-1}, y_{2n})$$

$$\therefore d(y_{2n}, y_{2n+1}) \le \lambda d(y_{2n-1}, y_{2n})$$

Hence the sequence $\{y_n\}_{n=0}^{\infty}$ is contractive. So, it is a Cauchy sequence in X. As X is a complete metric space, there exists a $z \in X$ such that $\lim_{n\to\infty} y_n = z$.

i.e. $\lim_{n\to\infty} y_{2n} = \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} = z$ and

$\lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = z$.

$\because S$ is continuous.

$\therefore \lim_{n\to\infty} SAx_{2n} = Sz$ and $\lim_{n\to\infty} S^2 x_{2n} = Sz$

$\because (A, S)$ are R-weakly commutating

$$\therefore |ASx_{2n} - Sz| \le |ASx_{2n} - SAx_{2n}| + |SAx_{2n} - Sz|$$
$$\le R|Sx_{2n} - Ax_{2n}| + |SAx_{2n} - Sz|$$

$\therefore \lim_{n\to\infty} ASx_{2n} = Sz$.

Similarly it may be proved that

$\therefore \lim_{n\to\infty} TBx_{2n+1} = Tz$ and $\lim_{n\to\infty} T^2x_{2n+1} = Tz \because T$ is continuous.

$\therefore \lim_{n\to\infty} BTx_{2n+1} = Tz$. $\because (B,T)$ are R -weakly commutating

Putting $x = Sx_{2n}$ and $y = x_{2n+1}$ in (1), we have

$$d\left(ASx_{2n}, Bx_{2n+1}\right) \le \lambda\ M(Sx_{2n}, x_{2n+1})$$

$$M(Sx_{2n}, x_{2n+1}) = \max\left\{\begin{array}{l} d\left(S^2x_{2n}, Tx_{2n+1}\right), d\left(ASx_{2n}, S^2x_{2n}\right), \\ d\left(Bx_{2n+1}, Tx_{2n+1}\right), \\ \frac{1}{2}\left(d\left(ASx_{2n}, Tx_{2n+1}\right) + d\left(S^2x_{2n}, Bx_{2n+1}\right)\right) \end{array}\right\}$$

Taking $\lim_{n\to\infty}$ to both sides, we get

$$\lim_{n\to\infty} d\left(ASx_{2n}, Bx_{2n+1}\right) \le \lambda \lim_{n\to\infty} \max\left\{\begin{array}{l} d\left(S^2x_{2n}, Tx_{2n+1}\right), d\left(ASx_{2n}, S^2x_{2n}\right), \\ d\left(Bx_{2n+1}, Tx_{2n+1}\right), \\ \frac{1}{2}\left(d\left(ASx_{2n}, Tx_{2n+1}\right) + d\left(S^2x_{2n}, Bx_{2n+1}\right)\right) \end{array}\right\}$$

$$d\left(Sz, z\right) \le \lambda \max\left\{\begin{array}{l} d\left(Sz, z\right), d\left(Sz, Sz\right), d\left(z, z\right), \\ \frac{1}{2}\left(d\left(Sz, z\right) + d\left(Sz, z\right)\right) \end{array}\right\}$$

$d\left(Sz, z\right) \le \lambda d\left(Sz, z\right) \Rightarrow Sz = z.$ …(3)

Similarly, it may be proved that $Tz = z.$ …(4)

Putting $x = z$ and $y = x_{2n+1}$ in (1), we have

$$d\left(Az, Bx_{2n+1}\right) \le \lambda\ M(z, x_{2n+1})$$

$$M(z, x_{2n+1}) = \max\left\{\begin{array}{l} d\left(Sz, Tx_{2n+1}\right), d\left(Az, Sz\right), \\ d\left(Bx_{2n+1}, Tx_{2n+1}\right), \\ \frac{1}{2}\left(d\left(Az, Tx_{2n+1}\right) + d\left(Sz, Bx_{2n+1}\right)\right) \end{array}\right\}$$

Taking $\lim_{n\to\infty}$ to both sides, we get

$$\lim_{n\to\infty} d\left(Az, Bx_{2n+1}\right) \le \lambda \lim_{n\to\infty} \max\left\{\begin{array}{l} d\left(Sz, Tx_{2n+1}\right), d\left(Az, Sz\right), \\ d\left(Bx_{2n+1}, Tx_{2n+1}\right), \\ \frac{1}{2}\left(d\left(Az, Tx_{2n+1}\right) + d\left(Sz, Bx_{2n+1}\right)\right) \end{array}\right\}$$

$\therefore d\left(Az, z\right) \le \lambda d\left(Az, z\right) \Rightarrow Az = z$ ...(5).

Similarly, it may be proved that $Bz = z.$ ...(6)

Thus from (3),(4),(5),(6) we get $Az = Bz = Sz = Tz = z.$ Thus z is a common fixed point of A, B, S, T .

<u>Uniqueness</u>: Let us try to prove that the fixed point is unique. Let if possible there are two fixed points say z and z^* of A, B, S, T . i.e. $Az = Bz = Sz = Tz = z$ and $Az^* = Bz^* = Sz^* = Tz^* = z^*$.

Put $x = z$ and $y = z^*$ in (1) , we have

$$d\left(Az, Bz^*\right) \le \lambda\ M(z, z^*)$$

where

$$M(z, z^*) = \max\left\{\begin{array}{l} d\left(Sz, Tz^*\right), d\left(Az, Sz\right), d\left(Bz^*, Tz^*\right), \\ \frac{1}{2}\left(d\left(Az, Tz^*\right) + d\left(Sz, Bz^*\right)\right) \end{array}\right\}$$

$$= \max\left\{\begin{array}{l} d\left(z, z^*\right), d\left(z, z\right), d\left(z^*, z^*\right), \\ \frac{1}{2}\left(d\left(z, z^*\right) + d\left(z, z^*\right)\right) \end{array}\right\} = d\left(z, z^*\right)$$

$$\therefore d\left(z, z^*\right) \le \lambda d(z, z^*) \Rightarrow z = z^*$$

Thus there exists a unique common fixed point for A, B, S, T .

3.1.2-Corrollary: Let A, S be self-maps of a complete metric space (X, d) .Suppose that S are continuous, the pair $\left(A, S\right)$ is R -weakly commutating and $A(X) \subseteq S(X)$. If there exists $\lambda \in [0,1)$ such that

$$d\left(Ax, Ay\right) \le \lambda\ M(x, y)$$

where $M(x, y) = \max\left\{\begin{array}{l} d\left(Sx, Sy\right), d\left(Ax, Sx\right), d\left(Ay, Sy\right), \\ \frac{1}{2}\left(d\left(Ax, Sy\right) + d\left(Sx, Ay\right)\right) \end{array}\right\}$

for $x, y \in X$ .Then there is a unique fixed point $u \in X$ such that $Au = Su = u.$

Proof: By substituting $A = B, S = T$ in Theorem 3.1.1, we get the proof.

Remark: Corollary-3.1.2 is the main result of Shehzad [5].

To prove the following result, theorem 3.1.1 is used.

The following result is a generalization of Jungk et. al. [1] with four mapping which in pair are R- sub weakly commutating..

3.1.3-Theorem: Let M be a non-empty q-star shaped subset of a normed space (X, d) and A, B, S, T are linear and continuous with $q \in F(S) \cap F(T)$ and $A(M) \subseteq T(M)$ and $B(M) \subseteq S(M)$. If the pairs (A, S) and (B, T) are R- sub weakly commutating for all $x.y \in M$

$$\left\|Ax, By\right\| \le \max\left[\begin{array}{l} \left\|Sx, Ty\right\|, dist\left(Sx, \left[Ax, q\right]\right), dist\left(Ty, \left[By, q\right]\right), \\ \frac{1}{2}\left\{dist\left(Sx, \left[By, q\right]\right) + dist\left(Ty, \left[Ax, q\right]\right)\right\} \end{array}\right]$$

...(1)

and $F(A) \cap F(B) \cap F(S) \cap F(T) \neq \phi$, provided one of the conditions holds:

(i) M is complete, cls $(A(M))$,cls $(B(M))$ is compact and A, B are continuous.

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(ii) M is weakly compact, $(T-A)$ is demi closed at 0 and X is complete.

(iii) M is weakly compact and X is a complete space satisfying Opial's condition.

Proof: Let us define $A_n : M \to M$ and $B_n : M \to M$ by

$A_n x = (1-k_n)q + k_n Ax$ and $B_n x = (1-k_n)q + k_n Bx$ ....(2)

for all $x \in M$ and a fixed sequence of real numbers $k_n (0 < k_n < 1)$ converging to 1. Then each A_n, B_n are self-mappings of M and for each $n \geq 1$, $A_n(M) \subseteq T(M)$ and $B_n(M) \subseteq S(M)$.

$\because$ T, S are linear , $A(M) \subseteq T(M)$ and $B(M) \subseteq S(M)$.

The linearity of S and R- subweakly commutating of (A,S) implies that

$$\|A_n Sx - SA_n x\| = \left\|\left((1-k_n)q + k_n ASx\right) - \left((1-k_n)q + k_n SAx\right)\right\|$$
$$\leq k_n \|ASx - SAx\| \leq k_n R\, dis\left[Sx, \left[Ax, q\right]\right]$$
$$\leq k_n R \|A_n x - Sx\| \text{ for all } x \in M .$$

This implies that the pair (A_n, S) is $k_n R$- weakly commutating for each $n \geq 1$. Similarly the pair (B_n, T) is $k_n R$- weakly commutating for each $n \geq 1$.

From (1), we get

$$\|A_n x, B_n y\| = k_n \|Ax - By\|$$
$$\leq k_n \max \begin{bmatrix} \|Sx, Ty\|, dist\left(Sx, \left[Ax, q\right]\right), dist\left(Ty, \left[By, q\right]\right), \\ \frac{1}{2}\left\{dist\left(Sx, \left[By, q\right]\right) + dist\left(Ty, \left[Ax, q\right]\right)\right\} \end{bmatrix}$$
$$\leq k_n \max \begin{bmatrix} \|Sx, Ty\|, \|Sx - A_n x\|, \|Ty - B_n y\|, \\ \frac{1}{2}\left\{\|Sx - B_n y\| + \|Ty - A_n x\|\right\} \end{bmatrix}$$

for all $x.y \in M$ and $0 < k_n < 1$.

By theorem-3.1.1 and for each $n \geq 1$ there exists $x_n \in M$ such that $Ax_n = Bx_n = Sx_n = Tx_n = x_n$

(i) The compactness of cls $(A(M))$ implies that there exists a subsequence $\{Ax_m\}$ of $\{Ax_n\}$ such that $Ax_m \to z$ as $m \to \infty$ .Then by (2)

$A_m x_m = (1-k_m)q + k_m Ax_m$

$\therefore$ as $m \to \infty$, $k_m \to 1$

$A_m x_m \to z$ implies $x_m \to z$ .Similarly by (2) i.e. definition of $B_m x_m \to z$ implies $x_m \to z$ .So by the continuity of S, T we have $z \in F(A) \cap F(B) \cap F(T) \cap F(S)$. Thus

$F(A) \cap F(B) \cap F(T) \cap F(S) \neq \phi$

(ii) Since M is weakly compact, there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to some $z \in M$ .But, S and T being linear and continuous are weakly continuous and the weak topology is Hausdorff , so we have $Sz = z = Tz$,and M is bounded, so

$$(S-A)x_m = Sx_m - Ax_m$$
$$= Sx_m - \frac{A_m x_m - (1-k_m)q}{k_m}$$
$$= \frac{k_m Sx_m - A_m x_m + (1-k_m)q}{k_m}$$
$$= \frac{k_m x_m - x_m + (1-k_m)q}{k_m}$$
$$= \frac{(k_m - 1)(x_m - q)}{k_m}$$
$$= \left(1 - k_m^{-1}\right)\left(x_m - q\right) \to 0 \text{ as } m \to \infty .$$

Now, the demiclosedness of $(S-A)$ at 0 guarantees that $(S-A)z = 0$.

Similarly we may prove that $(T-B)z = 0$ and hence $F(A) \cap F(B) \cap F(T) \cap F(S) \neq \phi$.

(iii) As in (ii), $Sz = z = Tz$ and $\|Sx_m - Ax_m\| \to 0$ as $m \to \infty$.

If $Sz \neq Bz$, then by Opial's condition

$$\liminf_{m\to\infty} \|Sx_m - Tz\| = \liminf_{m\to\infty} \|Sx_m - Sz\| < \liminf_{m\to\infty} \|Sx_m - Bz\|$$
$$\leq \liminf_{m\to\infty} \|Sx_m - Ax_m\| + \liminf_{m\to\infty} \|Ax_m - Bz\|$$
$$= \liminf_{m\to\infty} \|Ax_m - Bz\| \leq \liminf_{m\to\infty} \|Sx_m - Tz\|$$

this is a contradiction. Thus $Sz = Bz$ and hence $F(A) \cap F(B) \cap F(T) \cap F(S) \neq \phi$.

3.1.4-Corrollary: Let M be a non-empty q-star shaped subset of a normed space (X,d) and A, S, T are linear and continuous with $q \in F(S) \cap F(T)$ and $A(M) \subseteq T(M)$ and $A(M) \subseteq S(M)$ .If the pairs (A,S) and (A,T) are R- subweakly commutating and satisfy for all $x, y \in M$

$$\|Ax, Ay\| \leq \max \begin{bmatrix} \|Sx, Ty\|, dist\left(Sx, \left[Ax, q\right]\right), dist\left(Ty, \left[Ay, q\right]\right), \\ \frac{1}{2}\left\{dist\left(Sx, \left[Ay, q\right]\right) + dist\left(Ty, \left[Ax, q\right]\right)\right\} \end{bmatrix}$$

And $F(A) \cap F(S) \cap F(T) \neq \phi$, provided one of the conditions holds:

(i) M is complete, cls $(A(M))$,cls $(B(M))$ is compact and A, B are continuous.

(ii) M is weakly compact, $(T-A)$ is demi closed at 0 and X is complete.

(iii) M is weakly compact and X is a complete space satisfying Opial's condition.

Proof: Put $A = B$ in Theorem 3.1.3, we get the proof.

Remark: Corollary-3.1.4 is the main result of Jungk et.al. [1].

The following theorem is an extension of Bogin [2] using two mappings which in pair are weakly compatible.

3.1.5-Theorem: Let (X,d) be a metric space and $S, T : X \to X$ are mappings satisfying

$$d(Sx,Sy) \le a\max\left\{\begin{array}{l} d(Tx,Ty),d(Tx,Sx),d(Ty,Sy), \\ \frac{1}{2}\left[d(Tx,Sy)+d(Ty,Sx)\right] \end{array}\right\}$$

$$+b\left(d(Tx,Sx)+d(Ty,Sy)\right)+c\left(d(Tx,Sy)+d(Ty,Sx)\right) \quad \ldots.(1)$$

for all $x,y \in X$ where the real numbers $a,b,c>0$ satisfying the condition $a+2b+2c=1$, $S \subseteq T$, (S,T) are weakly compatible, then S,T have a unique common fixed point.

Proof: Let $x_0 \in X$.

$\because S \subseteq T$ and $S,T: X \to X$

$\therefore y_n = Sx_n = Tx_{n+1}$

Putting $x = x_n$ and $y = x_{n+1}$ in (1) we get,

$$d(Sx_n,Sx_{n+1}) \le a\max\left\{\begin{array}{l} d(Tx_n,Tx_{n+1}),d(Tx_n,Sx_n), \\ d(Tx_{n+1},Sx_{n+1}), \\ \frac{1}{2}\left[d(Tx_n,Sx_{n+1})+d(Tx_{n+1},Sx_n)\right] \end{array}\right\}$$

$$+b\left(d(Tx_n,Sx_n)+d(Tx_{n+1},Sx_{n+1})\right)$$

$$+c\left(d(Tx_n,Sx_{n+1})+d(Tx_{n+1},Sx_n)\right)$$

$$d(y_n,y_{n+1}) \le a\max\left\{\begin{array}{l} d(y_{n-1},y_n),d(y_{n-1},y_n),d(y_n,y_{n+1}), \\ \frac{1}{2}\left[d(y_{n-1},y_{n+1})+d(y_n,y_n)\right] \end{array}\right\}$$

$$+b\left(d(y_{n-1},y_n)+d(y_n,y_{n+1})\right)$$

$$+c\left(d(y_{n-1},y_{n+1})+d(y_n,y_n)\right)$$

$$d(y_n,y_{n+1}) \le a\max\left\{d(y_{n-1},y_n),d(y_n,y_{n+1}),\frac{1}{2}d(y_{n-1},y_{n+1})\right\}$$

$$+b\left(d(y_{n-1},y_n)+d(y_n,y_{n+1})\right)+c\left(d(y_{n-1},y_{n+1})+d(y_n,y_n)\right)$$

$$\because \frac{1}{2}d(y_{n-1},y_{n+1}) \le \frac{1}{2}\left(d(y_{n-1},y_n)+d(y_n,y_{n+1})\right)$$

$$\le \max\left\{d(y_{n-1},y_n),d(y_n,y_{n+1})\right\}$$

$$d(y_n,y_{n+1}) \le a\max\left\{d(y_{n-1},y_n),d(y_n,y_{n+1})\right\}$$

$$+b\left(d(y_{n-1},y_n)+d(y_n,y_{n+1})\right)+c\left(d(y_{n-1},y_{n+1})+d(y_n,y_n)\right)$$

Case-I: If $\max\left\{d(y_{n-1},y_n),d(y_n,y_{n+1})\right\} = d(y_n,y_{n+1})$ i.e.

$d(y_{n-1},y_n) < d(y_n,y_{n+1})$

$$d(y_n,y_{n+1}) \le a\max\left\{d(y_{n-1},y_n),d(y_n,y_{n+1})\right\}$$

$$+b\left(d(y_{n-1},y_n)+d(y_n,y_{n+1})\right)$$

$$+c\left(d(y_{n-1},y_{n+1})+d(y_n,y_n)\right)$$

$$d(y_n,y_{n+1}) \le ad(y_n,y_{n+1})+2bd(y_n,y_{n+1})+2cd(y_n,y_{n+1})$$

$$d(y_n,y_{n+1}) \le (a+2b+2c)d(y_n,y_{n+1}) \quad (\because a+2b+2c=1)$$

$$\therefore d(y_n,y_{n+1}) \le d(y_n,y_{n+1})$$

This is a contradiction.

Case-II: If $\max\left\{d(y_{n-1},y_n),d(y_n,y_{n+1})\right\} = d(y_n,y_{n-1})$

$$d(y_n,y_{n+1}) \le a\max\left\{d(y_{n-1},y_n),d(y_n,y_{n+1})\right\}+$$

$$b\left(d(y_{n-1},y_n)+d(y_n,y_{n+1})\right)$$

$$+c\left(d(y_{n-1},y_{n+1})+d(y_n,y_n)\right)$$

$$d(y_n,y_{n+1}) \le ad(y_{n-1},y_n)+b\left(d(y_{n-1},y_n)+d(y_n,y_{n+1})\right)$$

$$+c\left(d(y_{n-1},y_n)+d(y_n,y_{n+1})\right)$$

$$d(y_n,y_{n+1}) \le \frac{a+b+c}{1-c-b}d(y_n,y_{n-1}) \quad \left(\begin{array}{l} \because a+2b+2c=1 \\ \therefore \frac{a+b+c}{1-c-b}=1 \end{array}\right)$$

$\{d_n\} = \{d(y_n,y_{n-1})\}_{n=1}^{\infty}$ is a decreasing sequence …(2)

Putting $x = x_n$ and $y = x_{n+2}$ in (1) we get,

$$d(Sx_n,Sx_{n+2}) \le a\max\left\{\begin{array}{l} d(Tx_n,Tx_{n+2}),d(Tx_n,Sx_n), \\ d(Tx_{n+2},Sx_{n+2}), \\ \frac{1}{2}\left[d(Tx_n,Sx_{n+2})+d(Tx_{n+2},Sx_n)\right] \end{array}\right\}$$

$$+b\left(d(Tx_n,Sx_n)+d(Tx_{n+2},Sx_{n+2})\right)$$

$$+c\left(d(Tx_n,Sx_{n+2})+d(Tx_{n+2},Sx_n)\right)$$

$$d(y_n,y_{n+2}) \le a\max\left\{\begin{array}{l} d(y_{n-1},y_{n+1}),d(y_{n-1},y_n),d(y_{n+1},y_{n+2}), \\ \frac{1}{2}\left[d(y_{n-1},y_{n+2})+d(y_{n+1},y_n)\right] \end{array}\right\}$$

$$+b\left(d(y_{n-1},y_n)+d(y_{n+1},y_{n+2})\right)$$

$$+c\left(d(y_{n-1},y_{n+2})+d(y_{n+1},y_n)\right) \quad \ldots.(3)$$

$$\because d(y_{n-1},y_{n+1}) \le d(y_{n-1},y_n)+d(y_n,y_{n+1}) \le 2d(y_{n-1},y_n) \quad (4)$$

$$d(y_{n-1},y_{n+2})+d(y_{n+1},y_n) \le d(y_{n-1},y_n)+d(y_n,y_{n+2})+d(y_{n+1},y_n) \quad \ldots(5)$$

Putting (4) & (5) in (3), we get

$$d(y_n,y_{n+2}) \le a\max\left\{\begin{array}{l} 2d(y_{n-1},y_n),d(y_{n-1},y_n),d(y_{n+1},y_{n+2}), \\ \frac{1}{2}\left[d(y_{n-1},y_n)+d(y_n,y_{n+2})+d(y_{n+1},y_n)\right] \end{array}\right\}$$

$$+b\left(d(y_{n-1},y_n)+d(y_{n+1},y_{n+2})\right)$$

$$+c\left(d(y_{n-1},y_n)+d(y_n,y_{n+2})+d(y_{n+1},y_n)\right)$$

$$d(y_n,y_{n+2}) \le a\max\left\{\begin{array}{l} 2d(y_{n-1},y_n),d(y_{n-1},y_n),d(y_{n-1},y_n), \\ \frac{1}{2}\left[d(y_{n-1},y_n)+d(y_n,y_{n+2})+d(y_{n-1},y_n)\right] \end{array}\right\}$$

$$+b\left(d(y_{n-1},y_n)+d(y_{n-1},y_n)\right)$$

$$+c\left(d(y_{n-1},y_n)+d(y_n,y_{n+2})+d(y_{n-1},y_n)\right)$$

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$$d(y_n, y_{n+2}) \le a \max\left\{2d(y_{n-1}, y_n), \frac{1}{2}\left[2d(y_{n-1}, y_n) + d(y_n, y_{n+2})\right]\right\} + b\left(2d(y_{n-1}, y_n)\right) + c\left(2d(y_{n-1}, y_n) + d(y_n, y_{n+2})\right)$$

Case-III:

$$\max\left\{\begin{array}{l} 2d(y_{n-1}, y_n), d(y_{n-1}, y_n), \\ \frac{1}{2}\left[2d(y_{n-1}, y_n) + d(y_n, y_{n+2})\right] \end{array}\right\} = 2d(y_{n-1}, y_n)$$

$$\therefore d(y_n, y_{n+2}) \le 2ad(y_{n-1}, y_n) + 2bd(y_{n-1}, y_n) + c\left(2d(y_{n-1}, y_n) + d(y_n, y_{n+2})\right)$$

$$(1-c)d(y_n, y_{n+2}) \le (2a + 2b + 2c)d(y_{n-1}, y_n)$$

$\because a + 2b + 2c = 1$

$\therefore 1 + a + 2c < 2$

$$\therefore \frac{1+a}{1-c} < 2$$

$$\therefore d(y_n, y_{n+2}) \le \frac{1+a}{1-c} d(y_{n-1}, y_n)$$

$$\therefore d(y_n, y_{n+2}) \le k_1 d(y_{n-1}, y_n) \text{ where } k_1 = \frac{1+a}{1-c} < 2 \quad \text{....(6)}$$

Case-IV:

$$\max\left\{\begin{array}{l} 2d(y_{n-1}, y_n), d(y_{n-1}, y_n), \\ \frac{1}{2}\left[2d(y_{n-1}, y_n) + d(y_n, y_{n+2})\right] \end{array}\right\} = \frac{1}{2}\left[\begin{array}{l} 2d(y_{n-1}, y_n) \\ +d(y_n, y_{n+2}) \end{array}\right]$$

$$d(y_n, y_{n+2}) \le a\frac{1}{2}\left[2d(y_{n-1}, y_n) + d(y_n, y_{n+2})\right] + b\left(2d(y_{n-1}, y_n)\right) + c\left(2d(y_{n-1}, y_n) + d(y_n, y_{n+2})\right)$$

$$d(y_n, y_{n+2}) \le a\frac{1}{2}\left[2d(y_{n-1}, y_n) + d(y_n, y_{n+2})\right] + b\left(2d(y_{n-1}, y_n)\right) + c\left(2d(y_{n-1}, y_n) + d(y_n, y_{n+2})\right)$$

$$d(y_n, y_{n+2}) \le \frac{(2a + 4b + 4c)}{2 - a - 2c} d(y_{n-1}, y_n) = \frac{2}{1+2b} d(y_{n-1}, y_n)$$

Let $k_2 = \dfrac{2}{1+2b} < 2$

$$d(y_n, y_{n+2}) \le k_2 d(y_{n-1}, y_n) \quad \text{...(7)}$$

Let $k = \max\{k_1, k_2\}$ and $k < 2$

$$d(y_n, y_{n+2}) \le kd(y_{n-1}, y_n) \quad \text{...(8)}$$

Putting $x = x_{n+1}$ and $y = x_{n+2}$ in (1) we get,

$$d(Sx_{n+1}, Sx_{n+2}) \le a \max\left\{\begin{array}{l} d(Tx_{n+1}, Tx_{n+2}), d(Tx_{n+1}, Sx_{n+1}), d(Tx_{n+2}, Sx_{n+2}), \\ \frac{1}{2}\left[d(Tx_{n+1}, Sx_{n+2}) + d(Tx_{n+2}, Sx_{n+1})\right] \end{array}\right\} + b\left(d(Tx_{n+1}, Sx_{n+1}) + d(Tx_{n+2}, Sx_{n+2})\right) + c\left(d(Tx_{n+1}, Sx_{n+2}) + d(Tx_{n+2}, Sx_{n+1})\right)$$

$$d(y_{n+1}, y_{n+2}) \le a \max\left\{\begin{array}{l} d(y_n, y_{n+1}), d(y_n, y_{n+1}), d(y_{n+1}, y_{n+2}), \\ \frac{1}{2}\left[d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1})\right] \end{array}\right\} + b\left(d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2})\right) + c\left(d(y_n, y_{n+2}) + d(y_{n+1}, y_{n+1})\right)$$

From (2),(8), we get

$$d(y_{n+1}, y_{n+2}) \le a \max\left\{d(y_n, y_{n-1}), \frac{1}{2} kd(y_n, y_{n-1})\right\} + 2bd(y_n, y_{n-1}) + kcd(y_n, y_{n-1})$$

$\because k < 2$

$$\therefore \frac{1}{2} kd(y_n, y_{n-1}) < d(y_n, y_{n-1})$$

$$\therefore d(y_{n+1}, y_{n+2}) \le (a + 2b + kc)d(y_n, y_{n-1})$$

Let $\lambda = a + 2b + kc < a + 2b + 2c = 1$

$$d(y_{n+1}, y_{n+2}) \le \lambda d(y_{n-1}, y_n) \quad \text{...(9)}$$

$\because 0 \le \lambda < 1$

Therefore for any even integer $n \ge 0$ $d\left(y_{n+1}, y_{n+2}\right) \le \lambda^{n/2} d\left(y_0, y_1\right)$ and for odd integer $d\left(y_{n+1}, y_{n+2}\right) \le \lambda^{(n+1)/2} d\left(y_0, y_1\right)$

We get that $\{y_n\}$ is a Cauchy sequence. By the completeness of X there exists a $u \in X$ such that the sequence $\{y_n\}$ converges to u as $n \to \infty$.

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_{n+1} = u$$

As $\lim_{n\to\infty} y_n = u$ let there is $v \in X$ such that $Tv = u$ but $Sv \ne u$.

Now putting $x = x_n$ and $y = v$ in (1), we have

$$d(Sx_n, Sv) \le a \max\left\{\begin{array}{l} d(Tx_n, Tv), d(Tx_n, Sx_n), d(Tv, Sv), \\ \frac{1}{2}\left[d(Tx_n, Sv) + d(Tv, Sx_n)\right] \end{array}\right\} + b\left(d(Tx_n, Sx_n) + d(Tv, Sv)\right) + c\left(d(Tx_n, Sv) + d(Tv, Sx_n)\right)$$

Taking $\lim_{n\to\infty}$ to both sides, we get

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$$\lim_{n\to\infty} d(Sx_n, Sv)$$

$$\leq a \lim_{n\to\infty} \max\left\{\begin{matrix} d(Tx_n,Tv), d(Tx_n,Sx_n), d(Tv,Sv), \\ \frac{1}{2}\left[d(Tx_n,Sv)+d(Tv,Sx_n)\right] \end{matrix}\right\}$$

$$+b\lim_{n\to\infty}\left(d(Tx_n,Sx_n)+d(Tv,Sv)\right)$$

$$+c\lim_{n\to\infty}\left(d(Tx_n,Sv)+d(Tv,Sx_n)\right)$$

$$d(u,Sv) \leq a\max\left\{\begin{matrix} d(u,u), d(u,u), d(u,Sv), \\ \frac{1}{2}\left[d(u,Sv)+d(u,u)\right] \end{matrix}\right\}$$

$$+b\left(d(u,u)+d(u,Sv)\right)+c\left(d(u,Sv)+d(u,u)\right)$$

$$d(u,Sv) \leq (a+b+c)\,d(u,Sv)$$

$$\therefore (1-a-b-c)\,d(u,Sv) \leq 0 \Rightarrow Sv = u$$

$\therefore Sv = Tv = u$

$\because S,T$ are weakly compatible , we get.

$STv = TSv \Rightarrow Su = Tu$

Now putting $x = u$ and $y = v$ in (1), we have

$$d(Su,Sv) \leq a\max\left\{\begin{matrix} d(Tu,Tv), d(Tu,Su), d(Tv,Sv), \\ \frac{1}{2}\left[d(Tu,Sv)+d(Tv,Su)\right] \end{matrix}\right\}$$

$$+b\left(d(Tu,Su)+d(Tv,Sv)\right)+c\left(d(Tu,Sv)+d(Tv,Su)\right)$$

$$d(Su,u) \leq a\max\left\{\begin{matrix} d(Su,u), d(Su,Su), d(u,u), \\ \frac{1}{2}\left[d(Su,u)+d(u,Su)\right] \end{matrix}\right\}$$

$$+b\left(d(Su,Su)+d(u,u)\right)+c\left(d(Su,u)+d(u,Su)\right)$$

$$(1-a-c)\,d(Su,u) \leq 0 \Rightarrow Su = u$$

$\therefore Su = Tu = u$

Uniqueness: Let us try to show that the fixed point is unique. Let if possible there are two fixed points of S,T i.e. $Su = Tu = u$ and $Su^* = Tu^* = u^*$.

Now putting $x = u$ and $y = u^*$ in (1), we have

$$d(Su,Su^*) \leq a\max\left\{\begin{matrix} d(Tu,Tu^*), d(Tu,Su), d(Tu^*,Su^*), \\ \frac{1}{2}\left[d(Tu,Su^*)+d(Tu^*,Su)\right] \end{matrix}\right\}$$

$$+b\left(d(Tu,Su)+d(Tu^*,Su^*)\right)+c\left(d(Tu,Su^*)+d(Tu^*,Su)\right)$$

$$d(u,u^*) \leq a\max\left\{\begin{matrix} d(u,u^*), d(u,u), d(u^*,u^*), \\ \frac{1}{2}\left[d(u,u^*)+d(u^*,u)\right] \end{matrix}\right\}$$

$$+b\left(d(u,u)+d(u^*,u^*)\right)+c\left(d(u,u^*)+d(u^*,u)\right)$$

$$d(u,u^*) \leq (a+2c)\,d(u,u^*)$$

$$(1-a-2c)\,d(u,u^*) \leq 0 \Rightarrow u = u^*$$

Thus there exists a unique common fixed point of S and T .

3.1.6-Corollary: Let (X,d) be a metric space and $S: X \to X$ are mappings satisfying

$$d(Sx,Sy) \leq a\max\left\{\begin{matrix} d(x,y), d(x,Sx), d(y,Sy), \\ \frac{1}{2}\left[d(x,Sy)+d(y,Sx)\right] \end{matrix}\right\}$$

$$+b\left(d(x,Sx)+d(y,Sy)\right)+c\left(d(x,Sy)+d(y,Sx)\right)$$

for all $x, y \in X$ where the real numbers $a,b,c > 0$ .satisfying the condition $a+2b+2c = 1$ then S has a fixed point.

Proof: By substituting $S = T$ in Theorem-3.1.5, the proof is obtained.

Remark: The Corollary 3.1.6 is the extension of main result of Bogin [2].

Example-3.1.7: Let $(X,d) = R$. Let $Sx = \frac{x}{6}, Tx = \frac{x}{3}$, $a = \frac{3}{7}, b = \frac{1}{7}, c = \frac{1}{7}$

S,T are weakly compatible at 0.

$\because S(0) = T(0) \Rightarrow ST(0) = TS(0)$

$\therefore$ L.H.S= $d(Sx,Sy) = \frac{x}{6} - \frac{y}{6} = \frac{x-y}{6}$

$$\text{R.H.S.} = a\max\left\{\begin{matrix} d(Tx,Ty), d(Tx,Sx), d(Ty,Sy), \\ \frac{1}{2}\left[d(Tx,Sy)+d(Ty,Sx)\right] \end{matrix}\right\}$$

$$+b\left(d(Tx,Sx)+d(Ty,Sy)\right)+c\left(d(Tx,Sy)+d(Ty,Sx)\right)$$

$$= \frac{3}{7}\max\left\{\left|\frac{x}{3}-\frac{y}{3}\right|, \left|\frac{x}{6}-\frac{x}{3}\right|, \left|\frac{y}{6}-\frac{y}{3}\right|, \frac{1}{2}\left(\left|\frac{x}{6}-\frac{y}{3}\right|+\left|\frac{y}{6}-\frac{x}{3}\right|\right)\right\}$$

$$+\frac{1}{7}\left(\left|\frac{x}{6}-\frac{x}{3}\right|+\left|\frac{y}{6}-\frac{y}{3}\right|\right)+\frac{1}{7}\left(\left|\frac{x}{6}-\frac{y}{3}\right|+\left|\frac{y}{6}-\frac{x}{3}\right|\right)$$

$$= \frac{3}{7}\max\left\{\left|\frac{x-y}{3}\right|, \frac{x}{6}, \frac{y}{6}, \frac{1}{2}\left(\left|\frac{x-2y}{6}\right|+\left|\frac{y-2x}{6}\right|\right)\right\}$$

$$+\frac{1}{7}\left(\frac{x}{6}+\frac{y}{6}\right)+\frac{1}{7}\left(\left|\frac{x-2y}{6}\right|+\left|\frac{y-2x}{6}\right|\right)$$

$$= \frac{3}{7}\max\left\{\frac{x-y}{6}, \frac{x}{6}, \frac{y}{6}, \frac{1}{2}\left(\frac{2y-x}{6}+\frac{2x-y}{6}\right)\right\}$$

$$+\frac{1}{7}\left(\frac{x}{6}+\frac{y}{6}\right)+\frac{1}{7}\left(\frac{2y-x}{6}+\frac{2x-y}{6}\right) \quad \because y \leq x < 2y$$

$$= \frac{3}{7}\times\max\left\{\frac{x-y}{6}, \frac{x}{6}, \frac{y}{6}, \frac{x+y}{12}\right\}+\frac{x+y}{42}+\frac{3x+3y}{42}$$

$$= \frac{3x}{42}+\frac{x+y}{42}+\frac{3x+3y}{42} = \frac{7x+4y}{42}$$

$\therefore$ L.H.S.$\leq$ R.H.S.

Similarly we may prove that the result holds for $x \leq y < 2x$.

Thus all the conditions of theorem-3.1.5 are satisfied. The unique common fixed point is 0.

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REFERENCES

[1] G. Jungck, N. Hussain, "Common fixed point and invariant approximation results for non-commuting generalized (f, g)-non-expansive maps", J. Math. Anal. Appl. 321 ,2006, pp 851–861

[2] J. Bogin, "A generalization of a fixed point theorem of Goebel, Kirk and Shimi" , Canad. Math. Bull. 19 ,1976, pp 7–12.

[3] Lj. B Ciric, "On some discontinuous fixed point theorems in convex metric spaces" . Czechoslov. Math. J. 43(188),1993,pp 319-326

[4] M Gregus, "A fixed point theorem in Banach spaces", Boll. Unione Mat. Ital. Sez. A (5) 17 , 1980, pp 193–198.

[5] N. Shahzad, "Invariant approximations, generalized I-contractions, and R-Sub weakly commuting maps" , Fixed Point Theory and Applications 2005:1,(2005, pp 79–86.

[6] R. P. Pant, "Common fixed points of non-commuting mappings"' J. Math. Anal. Appl.188, No. 2, 1994, pp 436–440.

[7] Z.Opial, "Weak convergence of the sequence of successive approximations for non-expansive mappings", Bull. Amer. Math. Soc. 73, 1967, pp 591–597.