International Journal on Future Revolution in Computer Science & Communication Engineering ISSN: 2454-4248
Volume: 3 Issue: 10 198 – 201

Prime Γ– Radical and Radical TΓ – Ideal in Ternary Γ- Semirings

K. Sri Latha
Research Scholar, Department of Mathematics,
Rayalaseema University, Kurnool-518 002, India.
lathajay111@gmail.com

D. Madhusudhana Rao
Associate Professor , Department of Mathematics,
V.S.R & N.V.R. College , Tenali-522 202, India.
dmrmaths@gmail.com

G. Shobha Latha
Professor, Department of Mathematics,
Sri Krishnadeveraya University, Anantapur-515003, India.

Abstract— In this paper we investigate some important properties of prime Γ- radical of aTΓ-ideal in a ternary Γ– semiring. On some special properties of prime Γ–radical, radical TΓ-ideal are also obtain in the case when the ideals are k-TΓ-ideals and h-TΓ-ideals.

Keywords- *Ternary Γ–semiring, radicalTΓ-ideal, radical k-TΓ-ideal, radical h-TΓ– ideal*

*****

I. Introduction:

The notion of ternary Γ– semiring was introduced by M. SajaniLavanyaand D. MadhusudhanaRao in [5, 6] in the year 2015, as a natural generalization of ternary Γ– ring and Γ-semiring. The notion of prime radical of an ideal is important to the theory of semigroups, semirings as well as Γ-semigroups etc. In this paper we study prime Γ- radicals in ternary Γ- semiring as mentioned in the abstract.

II. Preliminaries:

Definition 2.1[5]: The non empty sets T and Γ together with a binary operation called addition and ternary multiplication denoted by juxtaposition is said to be a ***ternary Γ– semiring***if T and Γ be two additive commutative semigroups satisfying the following conditions.

(i) $[[x_1\alpha x_2\beta x_3]\gamma x_4\delta x_5] = [x_1\alpha\ [x_2\beta x_3\ \gamma x_4]\delta x_5] = [x_1\alpha x_2\beta\ [x_3\ \gamma x_4\delta x_5]]$

(ii) $[(x_1+x_2)\alpha x_3\beta x_4] = [x_1\alpha x_3\beta x_4] + [x_2\alpha x_3\beta x_4]$

(iii) $[x_1\alpha(x_2+x_3)\beta x_4] = [x_1\alpha x_2\beta x_4] + [x_1\alpha x_3\beta x_4]$

(iv) $[x_1\alpha x_2\beta(x_3+x_4)] =[x_1\alpha x_2\beta x_3] + [x_1\alpha x_2\beta x_4]$

for all $x_1,x_2,x_3,x_4,x_5\in$ Tand $\alpha,\beta,\gamma,\delta\in\Gamma$

Definition 2.2[5]: An element 0 in ternary Γ– semiring T such that $0 + a = a$ and $0\alpha a\beta b = a\alpha 0\beta b = a\alpha b\beta 0 = 0$ for all a, $b\in$T ,α, $\beta\in\Gamma$. Then 0 is called the ***0 – element*** or ***simply zero of the ternary Γ– semiring T.***

Definition 2.3[5]: An element a of a ternary Γ– semiring T is said to be an ***identity*** provided $a\alpha a\beta t = t\alpha a\beta a = a\alpha t\beta a$ = t for all t ∈ T, α, $\beta\in\Gamma$.

Definition 2.4[5]: A ternary Γ– semiring T is said to be ***commutative*** provided $a\alpha b\beta c = b\beta c\alpha a = c\alpha a\beta b = b\alpha a\beta c = c\alpha b\beta a = a\alpha c\beta b$ for all a, b, $c\in$ T, $\alpha,\beta\in\Gamma$.

Definition 2.5[5]: An additive subsemigroup S of T is said to be a ***ternary Γ– sub semiring*** provided $a\alpha b\beta c\in$ S for all a, b, $c\in$ S.

Definition 2.6[5]: An additive subsemigroup A of T is said to be a ***leftTΓ– ideal of T*** if$x\alpha y\beta a \in$ A for all $a\in$ A, $x,y\in$ T, α, $\beta\in\Gamma$.

Definition 2.7[5]: An additive subsemigroup A of T is said to be a ***lateralTΓ– ideal of T*** if$x\alpha a\beta y \in$ A for all $a\in$ A, $x,y\in$ T, α, $\beta\in\Gamma$.

Definition 2.8[5]: An additive subsemigroup A of T is said to be a ***rightTΓ– ideal of T*** if$a\alpha x\beta y \in$ A for all $a\in$ A, $x,y\in$ T, α, $\beta\in\Gamma$.

Definition 2.9]5]: An additive subsemigroup A of T is said to be a ***TΓ– ideal of T*** if$x\alpha y\beta a \in$ A, $x\alpha a\beta y \in$ A and $a\alpha x\beta y \in$ A.

Definition 2.10: A TΓ- ideal A of a ternary Γ- semiring T is said to be a ***k-TΓ-ideal*** if for $x,y\in$ T, $x+y\in$ A and $x\in$ A then $y\in$ A.

Definition 2.11: A TΓ- ideal A of a ternary Γ- semiring T is said to be a ***h-TΓ-ideal*** if for $x\in$ T, and for $a_1, a_2\in$ A, $x+ a_1 +t = a_2+t$, $t \in$ T implies $x\in$ A.

Definition 2.12[5]: Aproper TΓ- ideal P of a ternary Γ-semiring T is said to be a ***prime TΓ-ideal*** of T if for any three TΓ- ideals A, B, C of T, AΓBΓC ⊆ P implies A⊆P or B⊆P or C⊆P.

ISSN: 2454-4248
198 – 201

Definition 2.13 [6]: Aproper TΓ- ideal Q of T is said to be a ***semiprimeTΓ-ideal*** of T if AΓAΓA ⊆ Q implies A⊆Q for any TΓ- ideal A of T.

Definition 2.14: Anonempty subset M of a ternary Γ- semiring T is said to be an ***m– system*** if for each a, b, c∈ M, there exists elements $x_1,x_2,x_3,x_4$of T such that $a\Gamma x_1\Gamma b\Gamma x_2\Gamma c \subseteq$ M or $a\Gamma x_1\Gamma x_2\Gamma b\Gamma x_3\Gamma x_4\subseteq$ M or $a\Gamma x_1\Gamma x_2\Gamma b\Gamma x_3\Gamma c\Gamma x_4\subseteq$ M or $x_1\Gamma a\Gamma x_2\Gamma b\Gamma x_3\Gamma x_4\Gamma c\subseteq$ M.

III. Prime Γ – Radical of a TΓ- ideal:

Definition 3.1: Let T be a ternary Γ- semiring and A be a TΓ- ideal of T. Then ***prime Γ –Radical of A*** is denoted by rad(A) is defined to be the intersection of all prime TΓ- ideals of T each of which contains A.

Definition 3.2: A TΓ- ideal N in a ternary Γ- semiring T is said to be a ***nilpotent TΓ- ideal***if $(N\Gamma)^{2n}N = 0$ for some natural number n.

Theorem 3.3: In a ternary Γ– semiringTthe following conditions are equivalent.

(1) P is a prime TΓ- ideal of T.

(2) $a\Gamma T\Gamma b\Gamma T\Gamma c \subseteq P$, $a\Gamma T\Gamma T\Gamma b\Gamma T\Gamma T\Gamma c \subseteq P$, $a\Gamma T\Gamma T\Gamma b\Gamma T\Gamma c\Gamma T \subseteq P$,$T\Gamma a\Gamma T\Gamma b\Gamma T\Gamma T\Gamma c \subseteq P$ implies $a \in P$ or $b \in P$ or $c\in P$.

(3) $\langle a\rangle\Gamma\langle b\rangle\Gamma\langle c\rangle\subseteq P$ implies $a \in P$ or $b \in P$ or $c\in P$.

Corollary 3.4: A TΓ- ideal of A of a commutative ternary Γ– semiringT is prime if and only if $a\alpha b\beta c \in P$ implies $a \in P$ or $b \in P$ or $c\in P$ for all $a, b, c\in T$, $\alpha, \beta \in \Gamma$.

Theorem 3.5: For a TΓ- ideal A of a ternary Γ– semiringT we have the following.

(1) A ⊆ rad(A)

(2) If P is a prime TΓ- ideal of T then A ⊆ Piff rad(A)⊆ P

(3) If B is a TΓ- ideal of T satisfying A ⊆ B then rad(A) ⊆ rad(B)

(4) rad(A) is semiprimeTΓ- ideal of T.

(5) rad(A) = rad $[(A\Gamma)^{2n}$ A], n being an integer and n≥0.

(6) rad(A) contains every nilpotent TΓ- ideal of T.

(7) rad[rad(A)] = rad(A)

Proof: (1), (2), (3) follow immediately from the definition of prime Γ – radical.

4) Obviously rad(A) is a TΓ- ideal of T. Let CΓCΓC ⊆rad(A) where C is a TΓ- ideal of T. Now rad(A) = ∩{P_i / A ⊆P_i , P_i is a prime TΓ- ideal in T}. So CΓCΓC ⊆P_i, for all P_i . Then P_i being prime, C ⊆$P_i$for allP_i. Therefore C ⊆rad(A), proving raad(A) is a semiprimeTΓ- ideal of T.

5) Let A is a TΓ- ideal of T, $(A\Gamma)^{2n}A\subseteq$ A, where n is an integer and n≥0. Hence by condition (3) rad $[(A\Gamma)^{2n}A]\subseteq$ rad(A). Let a ∈rad(A). Now rad(A) = ∩{P_i / A ⊆P_i , P_i is a prime TΓ- ideal in T}. Then a∈P_i for all P_i . If possible let a ∉rad $[(A\Gamma)^{2n}A]$. Then there exist a prime TΓ- ideal Q in T such that $(A\Gamma)^{2n}A\subseteq$ Q and $a \notin$ Q. Now Q being prime, $(A\Gamma)^{2n}A\subseteq$ Q implies that A⊆ Q. Hence Q is some P_i . This gives a contradiction. Therefore a ∈rad $[(A\Gamma)^{2n}A]$. Consequently rad(A) = rad $[(A\Gamma)^{2n}$ A].

6) Let N be a nilpotent TΓ- ideal of T. Then $(N\Gamma)^{2n}N = \{0\}$, for some integer n≥0. Hence $(N\Gamma)^{2n}N \subseteq$ rad(A). So $(N\Gamma)^{2n}N$ ⊆$P_i$for allP_i containing A and P_i is a prime TΓ- ideal. Then N ⊆$P_i$for allP_i. Therefore N⊆rad(A).

Theorem 3.6: Let A be a TΓ- ideal in a ternary Γ– semiringT then rad(A) = {t∈T/every m-system in T which contains $t$has a non empty intersection with A}

Theorem 3.7: Let A be a TΓ- ideal in a ternary Γ– semiringT. If a∈rad(A) then there exist an integer n≥0 such that $(a\alpha)^{2n}a\in$ A for $\alpha \in \Gamma$.

Theorem 3.8: Suppose T is a commutative ternary Γ– semiring and M is an m – system in T containing a. Then there exist an integer n≥0 such that $(a\Gamma)^{2n}a\Gamma x\Gamma y \subseteq$ M where $x, y \in$ T.

Proof: Since a ∈ M, there existx_1,x_2,x_3,x_4∈ T such that $a\Gamma x_1\Gamma a\Gamma x_2\Gamma a\subseteq$ M or $a\Gamma x_1\Gamma x_2\Gamma a\Gamma x_3\Gamma x_4\Gamma a \subseteq$ M or $a\Gamma x_1\Gamma x_2\Gamma a\Gamma x_3\Gamma a\Gamma x_4\subseteq$ M or $x_1\Gamma a\Gamma x_2\Gamma a\Gamma x_3\Gamma x_4\Gamma a\subseteq$ M. It follows that $a\Gamma(x_1\Gamma a\Gamma x_2)\Gamma a\subseteq$ M ot T being commutative, $a\Gamma a\Gamma a\Gamma x_1\Gamma x_2\subseteq$ M or $a\Gamma a\Gamma a\Gamma x_1\Gamma x_2\Gamma x_3\Gamma x_4\subseteq$ M.

Let $a\Gamma a\Gamma a\Gamma x_1\Gamma x_2\subseteq$ M. Then there exist $x_5,x_6,x_7,x_8\in$ T such that $(a\Gamma)^4a\Gamma x_1\Gamma x_2\Gamma x_5\Gamma x_6\subseteq$ M or $(a\Gamma)^4a\Gamma x_1\Gamma x_2\Gamma x_5\Gamma x_6\Gamma x_7\Gamma x_8 \subseteq$ M. Let $a\Gamma a\Gamma a\Gamma x_1\Gamma x_2\Gamma x_3\Gamma x_4\subseteq$ M. Then there exist y_1, y_2,y_3 , $y_4\in$ T such that $(a\Gamma)^4a\Gamma x_1\Gamma x_2\Gamma x_3\Gamma x_4\Gamma y_1\Gamma y_2\subseteq$ M or $(a\Gamma)^4a\Gamma x_1\Gamma x_2\Gamma x_3\Gamma x_4\Gamma y_1\Gamma y_2\Gamma y_3\Gamma y_4\subseteq$ M.

Continuing in this way, we get for each integer n≥0 $(a\Gamma)^{2n}a\Gamma x\Gamma y\subseteq$ M for some $x, y\in$ T.

Theorem 3.9: Let A be a TΓ- ideal in a commutative ternary Γ– semiring T such that $(a\alpha)^n a\in$ A where $a\in$ T and $\alpha\in\Gamma$, n is a odd natural number then $a\in$ rad(A).

Proof: Let M be any m –system in T containing a . Then by theorem 3.8, $(a\alpha)^n x\beta y \in$ M for some $x,y\in$ T and $\alpha, \beta\in\Gamma$. As A is a TΓ- ideal and $(a\alpha)^n a$, $(a\alpha)^n x\beta y\in$ A for some odd natural number n. Therefore M∩A ≠ ϕ. Therefore by theorem 3.6, a ∈rad(A).

We can deduce the following theorem by combining theorem 3.7 and theorem 3.9.

Theorem 3.10: Suppose that T is a commutative ternary Γ– semiringand A is a TΓ- ideal of T. Then rad(A) = {a∈T/$(a\alpha)^{n-1}a\in$ A for some odd natural number n}.

Definition 3.11: A TΓ- ideal A in a ternary Γ– semiring T is calleda ***prime radical TΓ-ideal***if rad(A) = A.

Note: In this paper we simply called a prime radical TΓ- ideal to be a radical TΓ- ideal.

Theorem 3.12: If A is a TΓ- ideal in a ternary Γ– semiring T then the following are equivalent.

(1) rad(A) = A

(2) $(a\alpha)^{n-1}a\in$ A implies $a\in$ A for some odd natural number n.

Proof: (1) ⇒ (2): Let $(a\alpha)^{n-1}a \in$ A then by theorem 3.9, $a \in$ rad(A) = A.

(2) ⇒ (1): We know that A⊆rad(A). Let a ∈rad(A). By theorem 3.7, there exist an odd natural number n such that $(a\alpha)^{n-1}a \in$ A. Hence by hypothesis $a \in$ A. Hence rad(A) ⊆ A. Therefore rad(A) = A.

Definition 3.13: A k-TΓ- ideal in a ternary Γ– semiring T is said to be a ***radical k-TΓ- ideal*** provided it is a radical TΓ- ideal.

Definition 3.14: A h-TΓ- ideal in a ternary Γ– semiring T which also is a radical TΓ- ideal is called a ***radical h-TΓ- ideal***.

Theorem 3.15: Let A be a radical k-TΓ- ideal of a commutative ternary Γ– semiring T and P, Q be any two subsets of T then S = {x∈T/xΓPΓQ⊆ A} is a radical k-TΓ- ideal.

Proof: S is clearly a TΓ- ideal of T. Now, let $x + y\in$ S and $x\in$ S, $y\in$ T. Then $(x+y)\Gamma p\Gamma q\subseteq$ A and $x\Gamma p\Gamma q\subseteq$ A for all $p\in$ P and for all $q\in$ Q. So $y\Gamma p\Gamma q\subseteq$ A for all $p\in$ P and for all $q\in$ Q as A is a TΓ- ideal in T. Hence $y\in$ S.

Consequently, S is a k-TΓ- ideal in T. Let $(x\Gamma)^{n-1}x \in$ S for some odd natural number n, then $(x\Gamma)^{n-1}x\Gamma p\Gamma q \subseteq$ A for all $p\in$ P and for all $q\in$ Q which implies $((x\Gamma)^{n-1}x)\ \Gamma((p\Gamma)^{n-1}p)\Gamma((q\Gamma)^{n-1}q) \subseteq$ A for all $p\in$ P and for all $q\in$ Q as A is a TΓ- ideal in T. Therefore $(x\Gamma p\Gamma q\Gamma)^{n-1}x\Gamma p\Gamma q\subseteq$ A for all $p\in$ P and for all $q\in$ Q. So $x\Gamma p\Gamma q\subseteq$ A for all $p\in$ P and for all $q\in$ Q as A is a radical TΓ- ideal. Thus $x\Gamma P\Gamma Q\subseteq$ A and so $x\in$ S. Hence by theorem 3.12, S is also a radical TΓ- ideal.

Theorem 3.16: Let A be a radical h-TΓ- ideal of a commutative ternary Γ– semiring T and P, Q are two subsets of T then S = {x∈T/xΓPΓQ⊆ A} is a radical h-TΓ- ideal.

Proof: ClearlyS is aTΓ- ideal of T. Now, let $x\in$ T and $x+ a_1 +t = a_2+t$ for $t \in$ T and for $a_1, a_2\in$ S. Then$(x+ a_1 +t)\Gamma p\Gamma q$= $(a_2+t\ )\Gamma p\Gamma q$ for all $p\in$ P and for all $q\in$ Q. Therefore $x\Gamma p\Gamma q$ + $a_1\Gamma p\Gamma q$+ $t\Gamma p\Gamma q$= $a_2\Gamma p\Gamma q$+ $t\Gamma p\Gamma q$ where $t\Gamma p\Gamma q\subseteq$ T and $a_1\Gamma p\Gamma q\subseteq$ A, $a_2\Gamma p\Gamma q\subseteq$ A. So $x\Gamma p\Gamma q\subseteq$ A for all $p\in$ P and for all $q\in$ Q as A is a h-TΓ- ideal of T. Hence $x \in$ S.

Consequently, S is a h-TΓ- ideal. The proof of the part that S is a radical TΓ- ideal is similar to that in theorem 3.15.

7) By condition (1), A ⊆ rad(A). So by condition (3), rad(A) ⊆ rad[rad(A)]. Let $a\in$ rad[rad(A)] and $\{P_i\}_{i\in\Delta}$ be the family of prime TΓ- ideals of T such that A ⊆ P_i for alli∈Δ. Then by definition rad(A) ⊆ P_i for alli∈Δ. Hence rad[rad(A)] ⊆ P_i. Therefore $a\in$ P_i for alli∈Δ implies that $a\in$ rad(A). Therefore rad[rad(A)] = rad(A).

Theorem 3.17: In a ternary Γ– semiring intersection of any collection of radical TΓ- ideals is again a radical TΓ- ideal.

Definition 3.18: Suppose T is a ternary Γ– semiring with a ternary Γ–subsemiring A and a TΓ- ideal I, P = I∩A is a TΓ- ideal. If there is an another TΓ- idealJ such that I ⊆ J and P = J∩A, then we say I can be enlarged to be aTΓ- ideal in T which also contracts to P.

Theorem 3.19: Let A be an m – system and N be a TΓ- ideal of a ternary Γ– semiring T such that N∩A = ∅ then there exist a maximal TΓ- ideal M of T containing A such that M∩A = ∅ moreover M is a prime TΓ- ideal of T.

Theorem 3.20: Let T be a commutative ternary Γ– semiringand A be a ternary Γ– subsemiringof T. Let I be a radical TΓ- ideal of T such that $a\alpha b\beta c\in$ I, $a\in$ A, $b, c\in$ T, α, $\beta\in\Gamma$ imply either $a\in$ I or $b \in$ I or $c\in$ I. Then P = I∩A is a prime TΓ- ideal in A. Also I can be expressed as an intersection of prime TΓ- ideals each os which contracts to P.

Proof: Leta,b, $c\in$ A, α, $\beta\in$ Γ such that $a\alpha b\beta c\in$ P. Then $a\alpha b\beta c\in$ I.Therefore by hypothesis either $a\in$ I or $b \in$ I or $c\in$ I.Hence either $a\in$ P or $b \in$ P or $c\in$ P. So P becomes a prime TΓ- idealby corollary 3.4

Let X = ∩{J/J is a prime TΓ- ideal of T with I⊆ J and J∩A = P}. Then I⊆ X. To prove the reverse inclusion, let x ∉ I. Then the m – system M = $\{x\}\cup\{d\Gamma(x\Gamma)^{2n-1}x / d\in$ A but d ∉ P and n is a positive integer} has empty intersection with I. Then by theorem 3.19 there exist a maximal TΓ- ideal Q ⊇ I with Q∩M = ∅which is also prime.

Then P ⊆Q∩A.Again q∈Q∩A, $q\alpha x\beta x\in$ Q, Q being a TΓ- ideal of T. It follows that $q\alpha x\beta x \notin$ M. This together with definition of M and that $q\in$ A implies $q\in$ P. Therefore Q∩A ⊆ P. Hence P = Q∩A. Again x ∉ Q as $x\in$ M and M∩Q = ∅. Therefore x ∉ X and so X ⊆ I. Consequently, I = X.

IV. Conclusion:

In this paper mainly we studied about radical TΓ-ideals in ternary Γ-semirings.

ACKNOWLEDGMENT

The authors would like to thank the experts who have contributed towards preparation and development of the paper and the authors also wish to express their sincere thanks to the

referees for the valuable suggestions which lead to an improvement of this paper.

REFERENCES

[1] **MadhusudhanaRao. D.**,*Theory of* Γ*-ideal in*Γ*-semigroups* – Thesis, ANU (2011)**.**

[2] **MadhusudhanaRao. D, Anjaneyulu. A and GangadharaRao.A.**, *Prime* Γ*- Radicalsin* Γ*-Semigroups,* International e-journal of Mathematics and Engineering, 138 (2011), 1250-1259.

[3] **MadhusudhanaRao. D. and SrinivasaRao. G.,** *Prime Radicals and Completely Prime Radicals in Ternary Semirings,* International Journal of Basic Science and Applied Computing (IJBSAC) Volume-1, Issue 4, February 2015.

[4] **MadhusudhanaRao. D. and SrinivasaRao. G.,***Structure of P-Prime and P-Semiprime Ideals in Ternary Semirings-* International Journal of Innovation in Science and Mathematics(IJISM) Vol 3, Issue 2, February 2015, pp 24-31.

[5] **SajaniLavanya. M., MadhusudhanaRao. D., and V. Syam Julius Rajendra,***On Lateral Ternary Γ-Ideals of Ternary Γ-Semirings*-American International Journal of Research in Science, Technology, Engineering &Mathematics (AIJRSTEM), 12(1), September-November, 2015, pp: 11-14.

[6] **SajaniLavanya. M., MadhusudhanaRao. D., and V. Syam Julius Rajendra,** *Prime Bi-ternary****Γ****-Ideals in Ternary* ***Γ****-Semirings*-British Journal of Research, Volume 2, Issue 6, November-December, 2015, pp: 156-166.

[7] **Sarala. Y., Anjaneyulu.A. and MadhusudhanaRao. D.**, *Prime radicals in ternary semi groups,*International Organization Scientific Research Journal of Mathematics (IOSR- JM) ISSN: 2278-5728 Volume 4, Issue 5 (Jan.-Feb. 2013), pp 43-53.