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# Fuzzy Arithmetic and Extension Principle

Dr. R. Balakumar., M.Sc., M.Phil., Ph.D.,[1]
Associate Professor, Department of Mathematics,
Prist University, Thanjavur

M. Soundharya, [2]
M.Phil Research Scholar, Department of Mathematics,
Prist University, Thanjavur

***Abstract:-*** Fuzzy arithmetic is an extensively used instrument for dealing with uncertainty in a computationally competent method, recently and much better in the upcoming years.

This thesis aims to investigate the basic properties of fuzzy arithmetic as its title implies. The properties of fuzzy arithmetic definitions, examples are discussed. Here we investigates the properties of fuzzy sets, properties of fuzzy number, performing arithmetic operations on fuzzy number, properties of L-R fuzzy number, performing operations on L-R fuzzy number, properties of fuzzy interval and properties of L-R fuzzy interval.

Also, the extension principle and fuzzy arithmetic operations using extension principle are investigated. The fuzzy equation is solved by using the method of $\alpha - cut.$

***Keywords:*** *Fuzzy arithmetic, fuzzy set, fuzzy number, Extension principle and fuzzy equation.*

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## INTRODUCTION

According to Lofti A. Zadeh, the founder of fuzzy logic, a fuzzy set is a class of objects that has a continuum of membership grades. The set is characterized by membership functions which vary between zero and one.

"Fuzzy set theory is a marvellous tool for modelling the kind of uncertainty associated with vagueness, with Imprecision and with a lack of information regarding to a particular element of the problem at hand". Thus the idea of fuzziness is one of "enrichment" not of "replacement".

Zadeh's ideas have found application in computer science, artificial intelligence, decision analysis, information science, system science, control engineering, expert systems, pattern recognition, management science, operation research and robotics.The ideas of fuzzy set theory have been introduced in topology, abstract algebra, geometry graph and analysis.

Fuzzy logic as an approach in computer science uses the idea in which human brain thinks and solves problems. It uses the idea in which instead of using quantitative terms, natural language terms are used in order to approximate human decision making. For notions which depend upon their contexts, notions that are not able to be defined precisely, formally define fuzzy logic as a form of knowledge representation. It gives the ability to reason more like human in computerized devices.

In accordance with Zadeh's theory, fuzzy logic is a logic that may be considered as an effort to formalize two extraordinary human capabilities. First to all, fuzzy logic is functioning in imprecise and uncertain environment where it is dealt with partially true or partially possible cases, and these factors are taken into account to reason and to make rational decisions. Besides that, fuzzy logic has a capability to perform different tasks without making any computations and measurements.

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The mathematical concept of fuzzy logic is easy to understand. It is so flexible that given any system without starting again from scratch it is easy to layer on more functionality. It is tolerant to imprecise data. With fuzzy logic a fuzzy system can be created to match any data. The fuzzy system does not just replace but it also arguments and simplifies the implementation of a conventional control technique.

Another meaning of fuzzy logic is that it causes the source of confusion. In its narrow sense, the logical facet of fuzzy logic may be considered as a multivalued logical system to be described in a generalized form. The main task is to define how similar the above proposition to the vital principle of classical logic is.

In this thesis we see that standard arithmetic and algebraic operations, which are based after all on the foundations of classical set theory, can be extended to fuzzy arithmetic and fuzzy algebraic operations. In an application of a new fuzzy arithmetic to fuzzy calculus and fuzzy linear equations is discussed. Four computable operations on fuzzy numbers are given.

In the proof of convergence of the arithmetic mean to normal arithmetic mean is given. The constrained fuzzy arithmetic is formulated in nonstandard arithmetic form to overcome the disadvantages of standard fuzzy arithmetic that can't deal with constraints with linguistic variables. The basic properties of a proposed fuzzy arithmetic are defined, and the investigation is carried out for some common types of constrains.

The extension problem of Zadeg is an effective tool to develop fuzzy arithmetic. Sometimes the application of the extension principle that is very important to extend classical functions to fuzzy mappings is not convenient for making pessimistic decisions.This extension is accomplished with Zadeh's extension principle [Zadeh, 1975]. Fuzzy numbers, briefly described, are used here because such numbers are the basis for fuzzy arithmetic.

This dissertation work consists of same basic concepts of fuzzy set theory and fuzzy arithmetic operations. This dissertation also explains role played by fuzzy arithmetic operations in the concept of the Extension principle.

**PRELIMINARIES**

Fuzzy set theory was introduced by Zadeh in 1965.

**Definition 1.1**

The concept of a **fuzzy set** is an extension of the concept of a crisp set.

Just as a crisp set on a universal set U is defined by its characteristic function from U to {0,1}, a fuzzy set on a domain U is defined by its membership function from U to [0,1].

Let U be a non-empty set, to be called the universal set (or) the universe of discourse (or) simply a domain.Then by a fuzzy set on U is meant a function, $A : U \to [0,1]$. 'A' is called the **membership function**, A(x) is called the **membership grade** of x.

We also write,

$$A = \{x, A(x) : x \in U\}$$

**Example**

Let U = {1, 2, 3, 4, 5}.

A = The set of all numbers in U very close to 1.

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Where $A(x) = \frac{1}{x}$

$\Rightarrow$ A = {(1, 1), (2, 0.5), (3, 0.3), (4, 0.25), (5, 0.2)}

**Definition 1.2**

Let $\sigma$ and $\tau$ be two fuzzy sets of a set V. Then $\sigma$ is said to be **fuzzy subset** of $\tau$, written as $\sigma \subseteq \tau$, if $\sigma(x) \leq \tau(x)$ for every $u \in V$.

**Definition 1.3**

The **complement of A** is denoted by A' and it is defined as a fuzzy set on U for which,

$A'(x) = 1 - A(x)$ for every X in U.

**Definition 1.4**

Let A be a fuzzy set on U. Then the **Height of a fuzzy set A** is the largest membership grade obtained by any element of the set.

$$h(A) = \sup_{x \in X} A(x)$$

**Example**

(i) Let A = {(a, 0.2), (b, 0.3), (c, 0.8), (d, 0.9)}

Then $h(A) = 0.9$

(ii) If A is a non-empty crisp set then $h(A) = 1$.

**Definition 1.5**

Let A be a fuzzy set on U. Then A is said to be **normal fuzzy set**

if, $A(x) = 1$ for atleast one x in U.

**Example**

Aint

All non-empty crisp sets are normal.

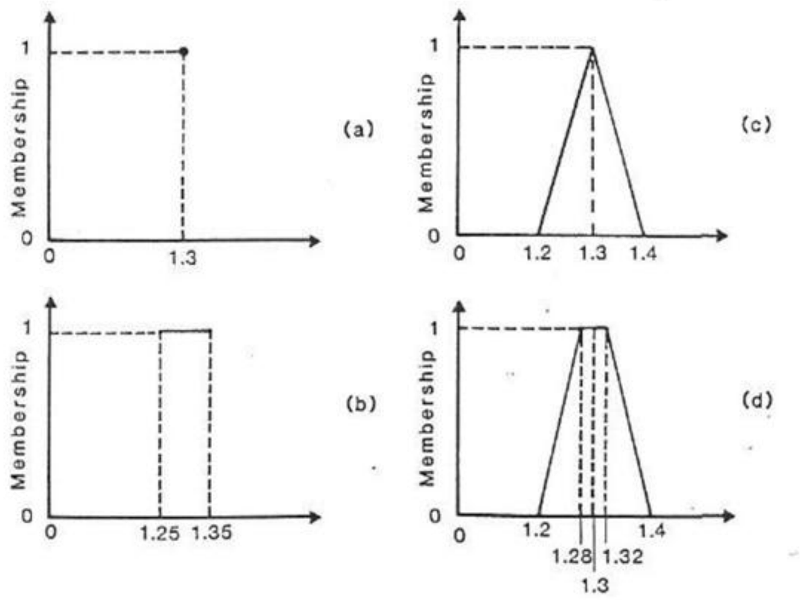
**FUZZY ARITHMETIC OPERATIONS**

**Definition 2.1**

Fuzzy numbers are the basis for fuzzy arithmetic. A fuzzy number is a fuzzy subset of the universe of numerical numbers.

**Example**

A fuzzy integer is a fuzzy subset of the domain of integers. while Fig. a is crisp number 1.3, depicts b the fuzzy number 1.3, or in other words the fuzzy set "around 1.3" or "close to 1.3". Fig. b,d are for the interval 1.25 to 1.35.




**Definition 2.2**

**Arithmetic operations on fuzzy number** defined by The $\alpha - cut$ of a fuzzy set and also each fuzzy number fully represent the fuzzy set and the fuzzy number; and for all $\alpha \in [0,1]$, $\alpha - cuts$ of each fuzzy number are considered as closed intervals of real numbers. Based on these properties, with respect to arithmetic operations on their $\alpha - cuts,$ fuzzy numbers arithmetic operations were made.

Before the arithmetic operations on fuzzy numbers, we recall some arithmetic operations on crisp interval.

**Definition 2.3**

**Crisp interval addition** is calculated in the form below with an example and figure illustrates this example.

$$[a,b]+[c,d]=[a+c,b+d]$$

$$[-2,1]+[0,5]=[-2,6]$$

**Example**

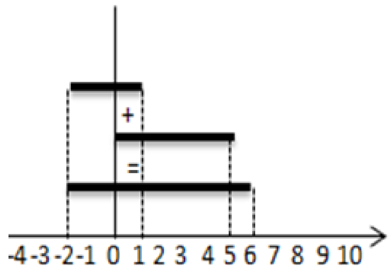



Figure: A crisp interval addition

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**Definition 2.4**

**Crisp interval subtraction** is calculated in the form below with an example and figure illustrates this example.

$$[a,b]-[c,d]=[a-c,b-d]$$

$$[-2,1]-[0,5]=[-7,1]$$

**Example**

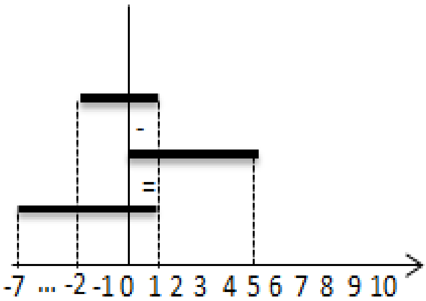


Figure: A crisp interval subtraction

**Definition 2.5**

**Crisp interval multiplication** is carried out in the form below with an example given.

$$[a,b]*[c,d]=[\min(ac,ad,bc,bd),\max(ac,ad,bc,bd)]$$

$$[2,4]*[3,2]=[\min(2\times 3,2\times 2,4\times 3,4\times 2),\max(2\times 3,2\times 2,4\times 3,4\times 2)]$$

$$=[\min(6,4,12,8),\max(6,4,12,8)]=[4,12]$$

**Definition 2.6**

**Crisp interval division** is carried out in the form below with an example given.

$$\frac{[a,b]}{[c,d]}=[a,b]\times\left[\frac{1}{d},\frac{1}{c}\right]=\left[\min\left(\frac{a}{c},\frac{a}{d},\frac{b}{c},\frac{b}{d}\right),\max\left(\frac{a}{c},\frac{a}{d},\frac{b}{c},\frac{b}{d}\right)\right]$$

$$\frac{[2,4]}{[1,2]}=[2,4]\times\left[\frac{1}{2},\frac{1}{1}\right]=\left[\min\left(\frac{2}{1},\frac{2}{2},\frac{4}{1},\frac{4}{2}\right),\max\left(\frac{2}{1},\frac{2}{2},\frac{4}{1},\frac{4}{2}\right)\right]$$

$$=[\min(2,1,4,2),\max(2,1,4,2)]=[1,4]$$

**3.L-R FUZZY NUMBER**

L-R type fuzzy number is a fuzzy number A described with membership function

$$\mu_A(x) = \begin{cases} 1, x \in [a,b] \\ R\left(\frac{x-b}{\beta}\right) \quad , x \in [b, b+\beta] \\ 0, otherwise \end{cases}$$

Denoted by $= (a,b,\alpha,\beta)_{LR}$, where A has its peak at $[a,b]$ with a and b the lower and upper modal values; L and R are the reference functions such that $[0,1] \rightarrow [0,1]$, with L(0)=R(0)=0.

Consider $L(t) = \frac{1}{1+t^2}, R(t) = \frac{1}{1+2|t|}, a = 6, \alpha = 3, \beta = 4$

Then

$$\mu_A(x) = \begin{cases} L\left(\frac{a-x}{\alpha}\right), x < 6 \\ 1, x = 6 \\ R\left(\frac{x-a}{\beta}\right), x > 6 \end{cases}$$

$$= \begin{cases} L\left(\frac{6-x}{3}\right), x < 6 \\ 1, x = 6 \\ R\left(\frac{x-6}{4}\right), x > 6 \end{cases}$$

$$= \begin{cases} \frac{1}{1+\left(\frac{6-x}{3}\right)^2}, x < 6 \\ 1, x = 6 \\ \frac{1}{1+2\left|\frac{x-6}{4}\right|}, x > 6 \end{cases}$$

## 4. EXTENSION PRINCIPLE IN FUZZY ARITHMETIC OPERATIONS

Extension principle is used as a mathematical instrument that takes crisp mathematical notions and procedures, extend them to the fuzzy realm, resulting in computable fuzzy sets by fuzzifying the parameters of a function.

Consider a fuzzy set A with the same inputs of function f.

$$y_1 = f(x_1), y_2 = f(x_2)..................y_n = f(x_n)$$

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + ....... + \mu_A(x_n)/x_n$$

Using extension principle, if the input of function f becomes fuzzy, there is a fuzzy output:

$$B = f(A) = \mu_A(x_1)/f(x_1) + \mu_A(x_2)/f(x_2) + ....... + \mu_A(x_n)/f(x_n)$$

Where each image of $x_i$ under function f becomes fuzzy with membership function $\mu_A(x_i)$. Many functions are many-to-one, which means if used, several x might map to a single y. The extension principle says that in such a case, in between the membership values of these several x elements of the fuzzy ser A, the maximum jshould be chosen as the membership of y. In the event where no element x in X is mapped to an output y, zero is the membership value of y element of set B.

$$\mu_B(y) = \max_{f^{-1}(y)} \mu_A(x)$$

**5. FUZZY EQUATIONS**

**EQUI-FUZZY**

A fuzzy set $A^{(\lambda)}$ over the support A is called Equi-fuzzy if for all $a \in A^{(\lambda)}, \mu_{A^{(\lambda)}}(a) = \lambda$ (membership of all the elements of $A^{(\lambda)}$ is $\lambda$) where $\lambda \in [0,1]$.

**SUPERIMPOSITION OF EQUI-FUZZY**

The following formula defines the operation of superimposition S over two equi-fuzzy sets $A^{(\lambda)}$ and $B^{(\delta)}$

$$A^{(\lambda)} S B^{(\delta)} = (A - A \cap B)^{(\lambda)} + (A \cap B)^{(\lambda+\delta)} (A - A \cap B)^{(\delta)}$$

Where $\lambda, \delta \geq 0, \lambda + \delta \leq 1$ and '+' denote the operation for union of disjoint fuzzy sets or otherwise.

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Given two fuzzy numbers A and B, the formula below defines the arithmetic operations using the method of $\alpha - cut.$

$$(A * B)^{\alpha} = A^{\alpha} * B^{\alpha}$$

where $A^{\alpha}, B^{\alpha}$ are $\alpha - cuts$ of A and B, $\alpha \in [0,1]$ and the '*' sign represents the arithmetic operation over A and B. In the situation of division where $0 \notin B^{\alpha}$ for any $\alpha \in [0,1]$. The subsequent fuzzy number A*B is represented as

$$A * B = (A * B)^{\alpha}.\alpha \qquad \text{……………(1)}$$

**SOLUTION OF THE FUZZY QUATION A+X=B BY USING THE METHOD OF $\alpha - cut$**

Given fuzzy equation A+X=B. Consider $A^{\alpha} = [a_1^{\alpha}, a_2^{\alpha}], B^{\alpha} = [b_1^{\alpha}, b_2^{\alpha}]$ and $X^{\alpha} = [x_1^{\alpha}, x_2^{\alpha}]$, to denote, respectively, the $\alpha - cuts$ of A,B and X. Then the equation A+X=B has solution if and only if the following properties hold:

1) for every $\alpha \in [0,1], b_1^{\alpha} - a_1^{\alpha} = b_2^{\alpha} - a_2^{\alpha}$
2) $\alpha < \gamma \Rightarrow b_1^{\alpha} - a_1^{\alpha} \le b_1^{\gamma} - a_1^{\gamma} \le b_2^{\gamma} - a_2^{\gamma} \le b_1^{\alpha} - a_1^{\alpha}$

The first property ensures the existence of the solution of the interval equation

$$A^{\alpha} + X^{\alpha} = B^{\alpha}$$

Which is

$$X^{\alpha} = [b_1^{\alpha} - a_1^{\alpha}, b_2^{\alpha} - a_2^{\alpha}]$$

The second property ensures that for $\alpha$ and $\gamma$ the solution of the interval equations are nested, i.e. if $\alpha \le \gamma$ then $X^{\alpha} \subseteq X^{\gamma}$. If for every $\alpha \in [0,1] X^{\alpha}$ has a solution and the second property is satisfied, then by (1) the fuzzy equation has solution X as

$$X = \bigcup_{\alpha \in [0,1]} X^{\alpha}.\alpha$$

**CONCLUSION**

Due to the benefits of membership grading in a fuzzy set, fuzzy arithmetic is recognized to have much influence than interval arithmetic. The linguistic terms can be used in the expression of fuzzy number making it feasible to compute with words instead of numbers. A significant property of fuzzy number is a closure under linear combination. This property eases computation with fuzzy numbers, and indicated the fuzzy number in a small number of parameters and leads to L-R fuzzy number.

In this thesis, we consider the properties of fuzzy numbers to be used to execute the arithmetic operations on them with reference to arithmetic operations on their $\alpha - cuts$

The Extension principle is used as a mathematical instrument that takes crisp mathematical notions and procedures, extend them to the fuzzy realm, resulting in computable fuzzy sets by fuzzifying the parameters of a function, and using the extension principle arithmetic operations on fuzzy numbers are performed. Some properties of fuzzy equation are discussed.

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